Topic 18

Composite Hypotheses

Examples

Outline

One-Sided test

Mark and Recapture

Two-Sided Test

Sample Proportion

The *p*-value

Mark and Recapture

Mark and recapture can be used as experimental procedure to test whether or not a population has reached a dangerously low level. The variables are

- t be the number captured and tagged,
- k be the number in the second capture,
- r the the number in the isecond capture that are tagged, and
- N be the total population.

If N_0 is the level that a wildlife biologist say is dangerously low, then the natural hypothesis is one-sided.

$$H_0 : N \geq N_0 \quad \text{versus} \quad H_1 : N < N_0.$$

Mark and Recapture

The data are used to compute r, the number in the second capture that are tagged. The likelihood function for N is the hypergeometric distribution,

$$L(N|r) = \frac{\binom{t}{r}\binom{N-t}{k-r}}{\binom{N}{k}}$$

The maximum likelihood estimate is $\hat{N} = [tk/r]$. Thus, higher values for r lead us to lower estimates for N. Let R be the (random) number in the second capture that are tagged, then, for an α level test, we look for the minimum value r_α so that

$$\pi(N) = P_N\{R \geq r_\alpha\} \leq \alpha \textit{ for all } N \geq N_0.$$

As N increases, then recaptures become less likely and the probability above decreases. Thus, we set the value of r_α according to the parameter value N_0, the minimum value under the null hypothesis.

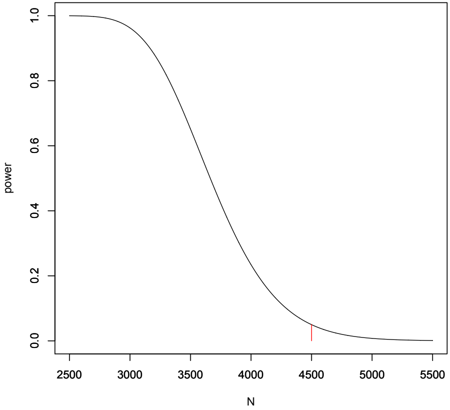
Mark and Recapture

To determine r_α for $\alpha = 0.05, 0.02, 0.01$,

```
> N0<-4500;t<-400;k<-500
> alpha<-c(0.05,0.02,0.01)
> ralpha<-qhyper(1-alpha,t,N0-t,k)
> data.frame(alpha,ralpha)
  alpha ralpha
1  0.05     54
2  0.02     57
3  0.01     59
```

The power curve for the case $\alpha = 0.05$ is given using the R commands

```
> N<-c(2500:5500)
> power<-1-phyper(54,t,N-t,k)
> plot(N,power,type="l",ylim=c(0,1))
```



Power curve. Vertical red segment at $N = N_0 = 4500$ has height $\alpha = 0.05$.

Mark and Recapture

Note that we must capture al least $r_\alpha = 54$ that were tagged in order to reject H_0 at the $\alpha = 0.05$ level. In this case the estimate for N is

$$\hat{N} = \left[\frac{kt}{r_\alpha}\right] = 3703$$

is well below $N_0 = 4500$.

Exercise. Determine the type II error rate for $N = 4000$ with

- $k = 800$ and $\alpha = 0.05$, 0.02, 0.01, and
- $\alpha = 0.05$ and $k = 600$, 800, and 1000.

Sample Proportion

Honey bees store honey for the winter. This honey serves both as nourishment and insulation from the cold. Typically for a given region, the probability of survival of a feral bee hive over the winter is $p_0 = 0.7$. To check whether this winter has a different survival probability, we consider the hypotheses

$$H_0 : p = p_0 \quad \text{versus} \quad H_1 : p \neq p_0.$$

If we use the central limit theorem, then, under the null hypothesis,

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}$$

has a distribution approximately that of a standard normal random variable. We reject if $|z|$ is too big.

Sample Proportion

For an α level test, the critical value is $z_{\alpha/2}$. The critical region

$$C = \left\{ \hat{p}; \left| \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} \right| > z_{\alpha/2} \right\}.$$

For this study, we examine 336 colonies and find that 250 survive.

Exercise. For $\alpha = 0.05$, determine whether or not we reject H_0.

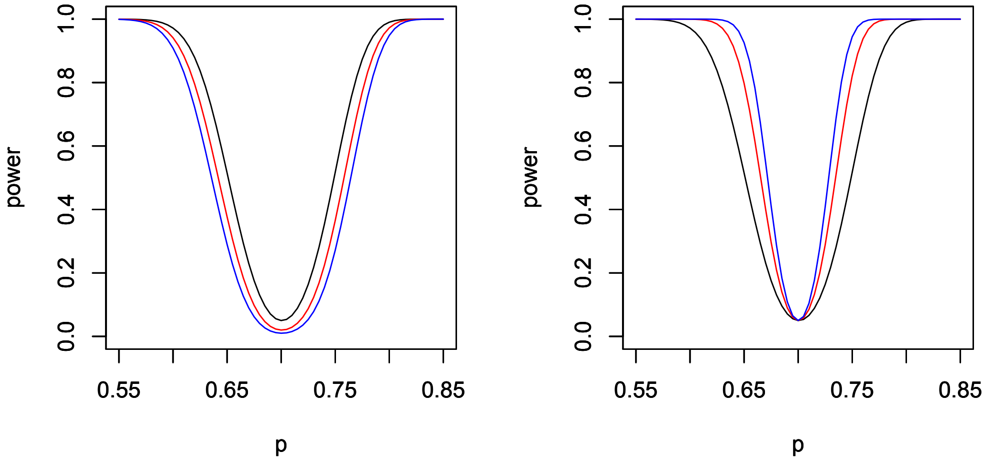
Exercise. Show that

$$-z_{\alpha/2} < \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} < z_{\alpha/2}$$

if and only if

$$\frac{p_0 - p}{\sqrt{p(1-p)/n}} - z_{\alpha/2}\sqrt{\frac{p_0(1-p_0)}{p(1-p)}} < \frac{\hat{p} - p}{\sqrt{p_(1-p)/n}} < \frac{p_0 - p}{\sqrt{p(1-p)/n}} + z_{\alpha/2}\sqrt{\frac{p_0(1-p_0)}{p(1-p)}}.$$

Sample Proportion



Power curve. (left) $n = 336$, $\alpha = 0.05, 0.02, 0.01$. (right) $\alpha = 0.05$, $n = 336, 672, 1008$.

The p-value

- The report of *reject* the null hypothesis does not describe the strength of the evidence because it fails to give us the sense of whether or not a small change in the values in the data could have resulted in a different decision.
- Consequently, one common method is to report the value of the test statistic and to give all the values for α that would lead to the rejection of H_0.
- The p-value is the probability of obtaining a result at least as extreme as the one that was actually observed, assuming that the null hypothesis is true. In this way, we provide an assessment of the strength of evidence against H_0.
- Consequently, a very low p-value indicates strong evidence against the null hypothesis.

The p-value

We can see how this works with the example on winter survival of beehives using the R command `prop.test`.

```
> prop.test(250,336,p=0.7)

1-sample proportions test with continuity correction

data:  250 out of 336, null probability 0.7
X-squared = 2.8981, df = 1, p-value = 0.08868
alternative hypothesis: true p is not equal to 0.7
95 percent confidence interval:
 0.6932518 0.7891515
sample estimates:
        p
0.7440476
```

The p-value states that we could reject H_0 for any significance level above this value.

The p-value

- Under the null hypothesis, $\hat{p}$ has approximately normal, mean $p_0 = 0.7$.
- The p-value, 0.089, is the area under the density curve outside the test statistic values

$$|z| = \left| \frac{\hat{p} - p_0}{p_0(1 - p_0)/n} \right| = 1.702$$

(indicated in blue),

- The critical value, 1.96, for an $\alpha = 0.05$ level test. (indicated in red).
- Because the p-value is greater than the significance level, we cannot reject H_0 at the 5% level.