Topic 19

Extensions on the Likelihood Ratio

One-Sided Tests

Outline

Introduction

For a composite hypothesis

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_1,$$

we have seen critical regions defined by taking a statistic $T(\mathbf{x})$ and defining the critical region based on a critical value $\tilde{k}_\alpha$. For a one-sided test, we have seen critical regions

$$\{\mathbf{x}; T(\mathbf{x}) \geq \tilde{k}_\alpha\} \quad \text{or} \quad \{\mathbf{x}; T(\mathbf{x}) \leq \tilde{k}_\alpha\}.$$

For a two-sided test, we saw

$$\{\mathbf{x}; |T(\mathbf{x})| \geq \tilde{k}_\alpha\}.$$

$\tilde{k}_\alpha$ is determined by the level α. We thus use commands `qnorm`, `qbinom`, or `qhyper` when $T(\mathbf{x})$ has, respectively, a normal, binomial, or hypergeometric distribution under a appropriate choice of $\theta \in \Theta_0$. We now examine extensions of the likelihood ratio test for simple hypotheses that have desirable properties for a critical region.

One-Sided Tests

In testing for the invasion of a mimic butterfly by a model species, we collected a simple random sample modeled as independent normal observations with unknown mean and known variance σ_0^2.

We discovered, in the case of a simple hypothesis test,

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu = \mu_1$$

that the critical region as determined by the Neyman-Pearson lemma depends only on whether or not μ_1 was greater than μ_0. For example, if $\mu_1 > \mu_0$, then the critical region

$$C = \{\mathbf{x}; \bar{x} \geq \tilde{k}_\alpha\}$$

shows that we reject H_0 whenever the sample mean is higher than some threshold value $\tilde{k}_\alpha$ *irrespective* of the difference between μ_0 and μ_1.

One-Sided Tests

- If a test is most powerful against *each* possible alternative in a simple hypothesis test, when we can say that this test is in some sense *best overall* for a composite hypothesis?
- Does this test have the property that its power function π is greater for every value of $\theta \in \Theta_1$ than the power function of *any* other test. Such a test is called uniformly most powerful.
- We can hope such a test if the procedures from simple hypotheses results in a common critical region for all values of the alternative.
- In the example above using independent normal data. In this case, the power function

$$\pi(\mu) = P_\mu\{\bar{X} \geq k_\alpha\}$$

increases as μ increases and so the test has the intuitive property of becoming more powerful with increasing μ.

Karlin-Rubin Theorem

In general, we look for a test statistic $T(\mathbf{x})$. Next, we check that the likelihood ratio,

$$\frac{L(\theta_2|\mathbf{x})}{L(\theta_1|\mathbf{x})}, \quad \theta_1 < \theta_2.$$

depends on the data $\mathbf{x}$ only through the value of statistic $T(\mathbf{x})$ and, in addition, this ratio is a monotone increasing function of $T(\mathbf{x})$.

The Karlinin-Rubin theorem states:

If these conditions hold, then for an appropriate value of $\tilde{k}_\alpha$,

$$C = \{\mathbf{x};\, T(\mathbf{x}) \geq \tilde{k}_\alpha\}$$

is the critical region for a uniformly most powerful α level test for the one-sided alternative hypothesis

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

Karlin-Rubin Theorem

A corresponding criterion holds for the one sided test a "less than" alternative.

Exercise. Verify that the likelihood ratio is an appropriate monotone function of the given test statistic, T.

1. For mark and recapture, use the hypothesis

$$H_0 : N \geq N_0 \quad \text{versus} \quad H_1 : N < N_0,$$

use the test statistic $T(\mathbf{x}) = r(\mathbf{x})$, the number tagged in the second capture.

2. For $X = (X_1, \ldots, X_n)$ is a sequence of Bernoulli trials with unknown success probability p, and the one-sided test

$$H_0 : p \leq p_0 \quad \text{versus} \quad H_1 : p > p_0,$$

use the test statistic $T(\mathbf{x}) = \hat{p}(\mathbf{x})$, the sample proportion of successes.

Binomial Test

If 20 out of 36 bee hives survive a severe winter, for an $\alpha = 0.05$ level test for

$$H_0 : p \geq 0.7 \quad \text{versus} \quad H_1 : p < 0.7,$$

we use the binomial distribution for the number of successes using `binom.test`.

```
> binom.test(20,36,p=0.7,alternative=c("less"))

Exact binomial test


data:  20 and 36
number of successes = 20, number of trials = 36, p-value = 0.04704
alternative hypothesis: true probability of success is less than 0.7
```

Exercise. Do we reject the hypothesis at the 5% level? the 1% level? Find the p-value using the `pbinom` command.

Proportion Test

If 250 out of 336 bee hives survive a mild winter, for an $\alpha = 0.05$ level test for

$$H_0 : p \leq 0.7 \quad \text{versus} \quad H_1 : p > 0.7,$$

we use the normal approximation for the number of successes using `prop.test`.

```
> prop.test(250,336,p=0.7,alternative=c("greater"))

1-sample proportions test with continuity correction

data:  250 out of 336, null probability 0.7
X-squared = 2.8981, df = 1, p-value = 0.04434
```

and we *reject* the null hypothesis.

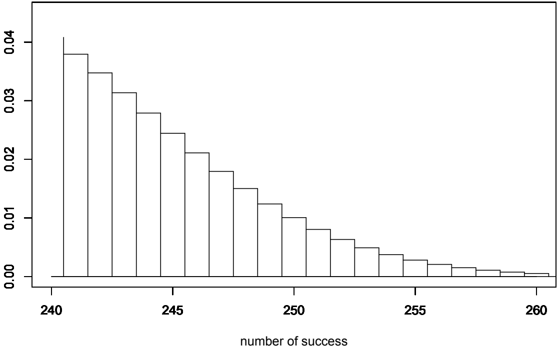
Continuity Correction

The p-value is $P\{X \geq 250\}$ where X is $Bin(336, 0.7)$. We compute this using R.

$$P\{X \geq 250\} = 1 - P\{X \leq 249\}$$

```
> 1-pbinom(249,336,0.7)
[1] 0.0428047
```

The command `prop.test` uses a normal approximation and a continuity correction to obtain a p-value 0.04434

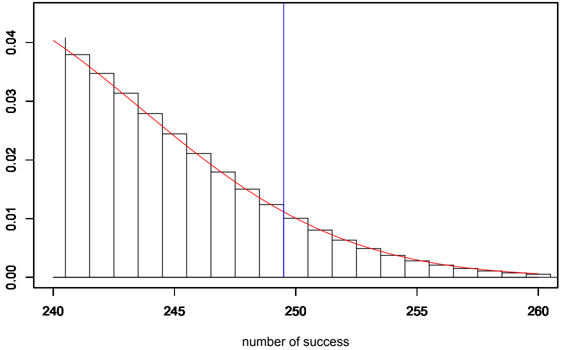


The p-value $P\{X \geq x\} = \sum_{y=x}^{n} P\{X = y\}$ can be realized as the area of rectangles, height $P\{X = y\}$ and width 1.

Continuity Correction

The rectangles look like a Riemann sum for the integral of the density of a $N(np_0, \sqrt{np_0(1-p_0)})$ random variable with lower limit $x - 1/2$.

```
> mu<-0.7*336
> sigma<-sqrt(336*0.7*0.3)
> x<-c(249,249.5,250)
> prob<-1-pnorm(x,mu,sigma)
> data.frame(x,prob)
      x       prob
1 249.0 0.05020625
2 249.5 0.04434199
3 250.0 0.03904269
```



The continuity correction replaces the binomial by finding the area under the normal density with lower limit $x - 1/2$.

Continuity Correction

Exercise.

1. Create a table of p-values for the hypothesis above for the values $x = 240, \ldots, 260$ for the number of hives that survive the winter. Compare the p-values using the binomial distribution and using the normal distribution with the continuity correction.
2. Use the normal approximation to create a 95% confidence interval for the proportion of hives that survive the winter.