Topic 20

t Procedures

One Sample Procedures

Outline

Introduction

The z-score is

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}.$$

taken under the assumption that the population standard deviation is known.
If we are forced to replace the unknown σ^2 with its unbiased estimator s^2, then the statistic is known as t:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}.$$

We have previously noted that for independent normal random variables the distribution of the t statistic can be determined exactly and we used the t distribution to construct a confidence interval for the population mean μ.

We now turn to using the t-statistic as a test statistic for hypothesis tests of the population mean. As with several other procedures we have seen, the two-sided t test is a likelihood ratio test.

Guidelines for Using the t Procedures

- Except in the case of small samples, the assumption that the data are a simple random sample from the population of interest is more important that the population distribution is normal.
- For sample sizes less than 15, use t procedures if the data are close to normal.
- For sample sizes at least 15 use t procedures except in the presence of outliers or strong skewness.
- The t procedures can be used even for clearly skewed distributions when the sample size is large, typically over 40 observations.

These criteria are designed to ensure that $\bar{x}$ is a sample from a nearly normal distribution. When these guidelines fail to be satisfied, then we can turn to alternatives that are not based on the central limit theorem, but rather use the rankings of the data.

One-Sample Tests

The two-sided hypothesis

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0,$$

based on independent normal observations $X_1, \ldots, X_n$ with unknown mean μ and unknown variance σ^2 is a likelihood ratio test. The parameter space and null hypothesis space, are, respectively,

$$\Theta = \{(\mu, \sigma^2); \mu \in \mathbb{R}, \sigma^2 > 0\} \quad \text{and} \quad \Theta_0 = \{(\mu, \sigma^2); \mu = \mu_0, \sigma^2 > 0\}.$$

The critical region is a level set for the t-statistic, $T(\mathbf{x})$ from the data $\mathbf{x}$.

$$C = \{|T(\mathbf{x})| > t_{n-1,\alpha/2}\}.$$

where $t_{n-1,\alpha/2}$ is the upper $\alpha/2$ tail probability of the t distribution with $n-1$ degrees of freedom.

One-Sample Tests

- Radon is formed as part of the normal radioactive decay chain of uranium.
- It is one of the densest substances that remains a gas under normal conditions.
- Radon gas from natural sources can accumulate in buildings, especially in confined areas such as attics, and basements.
- Epidemiological evidence shows a clear link between breathing high concentrations of radon and incidence of lung cancer.
    - According to the United States Environmental Protection Agency, radon is the second most frequent cause of lung cancer, after cigarette smoking, causing 21,000 lung cancer deaths per year in the United States.

One-Sample Tests

To check the reliability of radon detector, a university placed 12 detectors in a chamber having 105 picocuries of radon. (1 picocurie is 3.7×10^{-2} decays per second. This is roughly the activity of 1 picogram of radium 226.)

The two-sided hypothesis

$$H_0 : \mu = 105 \quad \text{versus} \quad H_1 : \mu \neq 105,$$

where μ is the mean value of the radon detectors. In other words, we are checking to see if the detector is biased either upward or downward.

The detector readings were:

91.9 97.8 111.4 122.3 105.4 95.0 103.8 99.6 96.6 119.3 104.8 101.7

One-Sample Tests

Using R, we find for an $\alpha = 0.05$ level significance test:

```
> mean(radon);sd(radon)
[1] 104.1333
[1] 9.397421
> qt(0.975,11)
[1] 2.200985
```

Thus, the t-statistic is

$$t = \frac{105 - 104.1333}{9.39742/\sqrt{12}} = -0.3195.$$

Thus, for a 5% significance test, $|t| < 2.200985$, the critical value and we *fail* to reject H_0.

One-Sample Tests

R handles this procedure easily.

```
> t.test(radon,alternative=c("two.sided"),mu=105)
One Sample t-test
data:  radon
t = -0.3195, df = 11, p-value = 0.7554
alternative hypothesis: true mean is not equal to 105
95 percent confidence interval:
  98.1625 110.1042
sample estimates:
mean of x
 104.1333
```

Exercise. Give an appropriate hypotheses for the case in which the concern is that the reading is too low. Repeat the `t.test` above and give the p-value.

Power Analysis

The power.t.test command considers five issues

- the sample size n,
- the difference between the null and a fixed value of the alternative delta,
- the standard deviation s,
- the significance level α, and
- the power $1-\beta$.

We can use power.t.test to drop out any one of these five and use the remaining four to determine the remaining value. For example, if we want to assure an 80% power against an alternative difference of 5 piconewtons,

```
> power.t.test(power=0.80,delta=5,sd=sd(radon),type=c("one.sample"))
```

The output shows that we need to make 30 measurements.

Power Analysis

Exercise. Fill in the tables based on the radon detector data set.

1. Consider a significance level $\alpha = 0.05$ and the standard deviation s from the data. Find the necessary number of observations.
2. Consider a significance level $\alpha = 0.05$ and a difference delta of 5 piconewtons. Find the power.

	power		
delta	0.7	0.8	0.9
5			
10			

	observations		
s	10	20	30
5.00			
9.40			

3. Which two values in each table are most extreme? Explain why.

Correspondence between Two-Sided Tests and Confidence Intervals

For a two-sided t-test, we have the following list of equivalent conditions:

fail to reject with significance level α.

$$\left|\frac{\mu_0 - \bar{x}}{s/\sqrt{n}}\right| = |t| < t_{n-1,\alpha/2}$$

$$-t_{n-1,\alpha/2} < \frac{\mu_0 - \bar{x}}{s/\sqrt{n}} < t_{n-1,\alpha/2}$$

$$-t_{n-1,\alpha/2}\frac{s}{\sqrt{n}} < \mu_0 - \bar{x} < t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}$$

$$\bar{x} - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}} < \mu_0 < \bar{x} + t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}$$

μ_0 is in the $\gamma = 1 - \alpha$ confidence interval