Assignment 4
Probability and Estimation

Math 363

November, 2009

1. Consider the function $g(x) = 1/(1 + x^3)$.
   (a) Estimate the integral
   $$\int_0^2 g(x)\, dx$$
   using Monte Carlo simulations with 100 uniform random variable on $[0, 2]$.
   (b) Repeat the estimate 100 times and estimate the standard deviation of the error.
   (c) If the standard deviation of the error is $\sigma/\sqrt{n}$ for some σ use parts (a) to estimate σ.
   (d) Repeat the estimate 100 times using 900 uniform random variables and check your estimate of the error with the estimate based on using 100 samples.
2. Let $X_1, X_2 \ldots, X_{60}$ be independent discrete random variables with mass function
   $$f_X(x) = x/15, \quad x = 1, 2, 3, 4, 5.$$
   (a) Let $\bar{X} = (X_1 + X_2 + \cdots + X_{60})/60$. Describe the approximating normal random variable.
   (b) Simulate $\bar{X}$ 1000 times and compare the mean and standard deviation with the result in part (b).
   (c) Estimate $P\{\bar{X} < 3.5\}$ using the central limit theorem and compare this to the value given by the simulation.
3. The goal is to measure the probability that a feral bee hive survives a winter. For this study, 112 colonies have been chosen.
   (a) If $p = 0.7$ is the probability of survival, give the mean and the standard deviation of an approximating normal random variable for the number of surviving colonies.
   (b) Use this to estimate the probability that fewer than 80 colonies survive and the probability that more than 90 survive.
   (c) If the study is suspended for a year and we perform the study for two year survival. If survival is independent from one winter to the next, give the probability q that the hive survives 2 years.
   (d) We can see that $p = \sqrt{q}$. Let $\hat{q}$ be the observed fraction of hives that survive for two years. Use the delta method to give a normal approximation to $\hat{p}$ based on the knowledge of $\hat{q}$.

4. The focal length f of an optical instrument is needed. This is determined by using the thin lens formula,

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{f}.$$

where s_1 is the distance from the lens to the object and s_2 is the distance from the lens to the real image of the object.

The distances s_1 and s_2 are each independently measured 25 times. The mean of the measurements is the actual distances, 12 centimeters and 15 centimeters, respectively. The standard deviation of the measurement is 0.1 centimeters for s_1 and 0.5 centimeters for s_2.

(a) Let $\bar{S}_1$ be the sample mean of the 25 measurements to the object. Estimate, using the central limit theorem, $P\{\bar{S}_1 > 12.01\text{cm}\}$.

(b) Let $\bar{S}_2$ be the sample mean of the 25 measurements to the image. Estimate, using the central limit theorem, $P\{\bar{S}_2 > 15.01\text{cm}\}$.

(c) How many measurements are needed so that $P\{|\bar{S}_2 - 15\text{cm}| > 0.01\text{cm}\} \leq 0.05$.

(d) Estimate the focal length using

$$\frac{1}{\bar{S}_1} + \frac{1}{\bar{S}_2} = \frac{1}{F}.$$

Use the delta method to give an estimate of the standard deviation of F.

5. The independent random variables $X_1, X_2, \ldots, X_n$ have common cumulative probability distribution function

$$F_X(x|\alpha, \beta) = \begin{cases} 0 & \text{if } x < 0, \\ (x/\beta)^\alpha & \text{if } 0 \leq x < \beta, \\ 1 & \text{if } x \geq \beta. \end{cases}$$

(a) Find the density of these random variables.

(b) The maximum likelihood estimator $\hat{\beta}$ for β is the maximum value in the data. Find the maximum likelihood estimators for α.

(c) The length (in millimeters) of cuckoos' eggs found in hedge sparrow nests can be modeled with this distribution. For the data

22.0	23.9	20.9	23.8	25.0	24.0	21.7	23.8	22.8	23.1	23.1	23.5	23.0	23.0

give the maximum likelihood estimators.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample with probability density function

$$f_X(x|\theta) = \theta x^{\theta-1}, \quad 0 \leq x \leq 1, \quad \theta > 0.$$

(This is a $Beta(\theta, 1)$ distribution.)

(a) Find the maximum likelihood estimator for θ.

(b) Find the method of moments estimator of θ.

(c) Use simulation to check the standard deviation of these estimators for $\theta = 1/2$ and $\theta = 2$.