Name ____________________

Mathematics 564 - Theory of Probability

Final Exam

December 16, 2008

Do any 7.

1. A bag has 9 blocks - 6 are green and 3 are blue.

   (a) Give the probability that the first two blocks are green.

   (b) For events A_1 and A_2, give a mathematical expression for the property "A_1 and A_2 are independent."

   (c) Let A_i be the event, the i-th block chosen is green. Are A_1 and A_2 independent? Explain your answer using the definition above.

   (d) Choose 4 *without replacement* and let X be the number of green blocks? What is $P\{X = 3\}$?

(e) Choose 4 *with replacement* and again let X be the number of green blocks. Find $P\{X = 3\}$.

2. Flip a coin, with heads turn up one card, with tails, turn up two. Let X be the number of $\heartsuit$s

   (a) Find $P\{X = x|\text{coin turns up heads}\}$.

   (b) Find $P\{X = x\}$ for $x = 0, 1, 2$.

   (c) Find $P\{\text{coin turns up tails}|X = 0\}$.

   (d) Find $P\{\text{coin turns up heads}|X = 0\}$.

3. For x and y taking on the values 0, 1, and 2, define

$$f_{Y|X}(y|x) = c_x(x + y)$$

for constants $c_0 = 1/3$ and $c_1 = 1/6$.

(a) Find c_2.

(b) Compute $E[Y^2|X = 1]$.

(c) Give a table for the joint mass function for X and Y when $f_X(x) = x/3$.

(d) Find $P\{X = Y\}$.

(e) Find the marginal mass function f_Y.

4. Let X have density

$$f_X(x) = \begin{cases} \frac{3}{8}(1-x)^2 & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Give the cumulative distribution function for X.

(b) Find EX.

(c) Find $\mathrm{Var}(X)$.

(d) Find the density of $Y = X^2$.

5. Let X be an exponential random variable with density

$$f_X(x) = e^{-x/2}.$$

(a) Find the moment generating function M_X for X.

(b) Let $X_i,\ i = 1, 2, 3,$ be independent random variables with density f_X Find the moment generating function for the sum $S = X_1 + X_2 + X_3$.

(c) Use the moment generating function to find the mean of S.

(d) Use the moment generating function to find the variance of S.

(e) Find the variance $\mathrm{Var}(3 - 2S)$.

6. Let X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} cx^2 + y^2 & \text{if } 0 \leq x \leq 2, 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the constant c.

(b) FInd the marginal density for X.

(c) Find the conditional density $f_{Y|X}(y|x)$.

(d) Find the conditional mean $E[Y|X]$.

7. Let$X$ and Y be independent standard normal random variables. Thus,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2},$$

(a) Set $(U, V) = g(X, Y)$ where

$$U = \frac{3}{5}X + \frac{4}{5}Y, \qquad U = \frac{4}{5}X - \frac{3}{5}Y.$$

(b) Give the inverse transformation $(X, Y) = g^{-1}(U, V)$ and find the Jacobian of this transformation.

(c) Find the joint density of U and V.

(d) Are U and V independent? Explain your answer.

8. Assume that the lifetime of a lightbulb has mean $\mu = 1000$ hours with a standard deviation of $\sigma = 200$ hours.

   (a) State the central limit theorem for $\bar{X}$, the average lifetime of n lightbulbs.

   (b) Use this to approximate the probability that for 100 observations, $\bar{X} > 1050$.

   (c) We now introduce a second lightbulb with is hypothesized to have the same properties as the first. We check this by looking at the difference between $\bar{X} - \bar{Y}$ for the two bulbs. Based on 100 observations of each type of lightbulb, find the mean and variance of $\bar{X} - \bar{Y}$.

   (d) Approximate, using the central limit theorem, $P\{|\bar{X} - \bar{Y}| > 50\}$.