Topic 8: Examples of Mass Functions and Densities

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We will describe several parameterized families of random variables indexed by some parameter θ and write P_θ to indicate the parameter value used to compute probabilities,

1 Examples of Discrete Random Variables

For discrete random variables, we write

$$f_X(x|\theta) = P_\theta\{X = x\}.$$

1. (Bernoulli) $Ber(p)$, $D = \{0, 1\}$

$$f_X(x|p) = p^x(1-p)^{1-x}.$$

2. (binomial) $Bin(n, p)$, $D = \{0, 1, \ldots, n\}$

$$f_X(x|p) = \binom{n}{x} p^x(1-p)^{n-x}.$$

So $Ber(p)$ is $Bin(1, p)$.

3. (geometric) $Geo(p)$, $D = \mathbb{N}$

$$f_X(x|p) = p(1-p)^x.$$

This random variable is the number of failing Bernoulli trials before the first success.

4. (negative binomial) $Negbin(n, p)$, $D = \mathbb{N}$

$$f_X(x|p) = \binom{n+x-1}{x} p^n(1-p)^x.$$

This random variable is the number of failing Bernoulli trials before the n-th success. Thus, $Geo(p)$ is $Negbin(1, p)$. Looking at the number of trials between successive successes, we see that $Negbin(n, p)$ is the sum of n independent $Geo(p)$ random variables.

To find the mass function, note that for X to take on a given value x, we must have $n-1$ successes in $n+x-1$ Bernoulli trials and success in the last trial. These two events are independent and so their probabilities multiply.

$$\begin{aligned} P_p\{X = x\} &= P_p\{n-1 \text{ successes in } n+x-1 \text{trials}\}P_p\{\text{success in the } n-x\text{-th rial}\} \\ &= \binom{n+x-1}{n-1} p^{n-1}(1-p)^{x-1} \cdot p = \binom{n+x-1}{x} p^n(1-p)^{x-1} \end{aligned}$$

5. (Poisson) $Pois(\lambda)$, $D = \mathbb{N}$,

$$f_X(x|\lambda) = \frac{\lambda^x}{x!} e^{-\lambda}.$$

The Poisson distribution approximates of the binomial distribution when n is large, p is small, but the product $\lambda = np$ is moderate.

$$\begin{aligned}
P\{X=0\} &= \tbinom{n}{0}p^0(1-p)^n &&= \left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda} \\
P\{X=1\} &= \tbinom{n}{1}p^1(1-p)^{n-1} &&= n\frac{\lambda}{n}\left(1-\frac{\lambda}{n}\right)^{n-1} \approx \lambda e^{-\lambda} \\
&\vdots &&\vdots \qquad \vdots \\
P\{X=x\} &= \tbinom{n}{x}p^x(1-p)^{n-x} &&= \frac{(n)_x}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} \approx \frac{\lambda^x}{x!}e^{-\lambda}.
\end{aligned}$$

6. (uniform) $U(a,b)$, $D = \{a, a+1, \ldots, b\}$,

$$f_X(x|a,b) = \frac{1}{b-a+1}.$$

Thus each value in the designated range has the same probability.

Exercise 1. *Check that* $\sum_{x\in D} f_X(x|\theta) = 1$ *in the examples above.*

2 Examples of Continuous Random Variables

1. (uniform) $U(a,b)$ on $[a,b]$,

$$f_X(x|a,b) = \frac{1}{b-a}.$$

2. (exponential) $Exp(\lambda)$ on $[0,\infty)$,

$$f_X(x|\lambda) = \lambda e^{-\lambda x}.$$

If we consider Bernoulli trials at a rate of n trials per time unit and take the approximation seen in the Poisson random variable, then the time of the first success

$$P\{T > t\} = (1-p)^{nt} = \left(1-\frac{\lambda}{n}\right)^{nt} \approx e^{-\lambda t}$$

The cumulative distribution function

$$F_T(t) \approx 1 - e^{-\lambda t}.$$

The density can be found by takng a derivative.

3. (gamma) $\Gamma(\alpha,\beta)$ on $[0,\infty)$,

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}.$$

Observe that $Exp(\lambda)$ is $\Gamma(1,\lambda)$. A $\Gamma(n,\lambda)$ can be seen as an approximation to the negative binomial random variable using the ideas that leads from the geometric random variable to the exponential. Alternatively, $\Gamma(n,\lambda)$ is the sum of n independent $Exp(\lambda)$ random variables. The **gamma function**

$$\Gamma(s) = \int_0^\infty x^s e^{-x}\,\frac{dx}{x}$$

is computed in R using `gamma(s)`.

For the density,

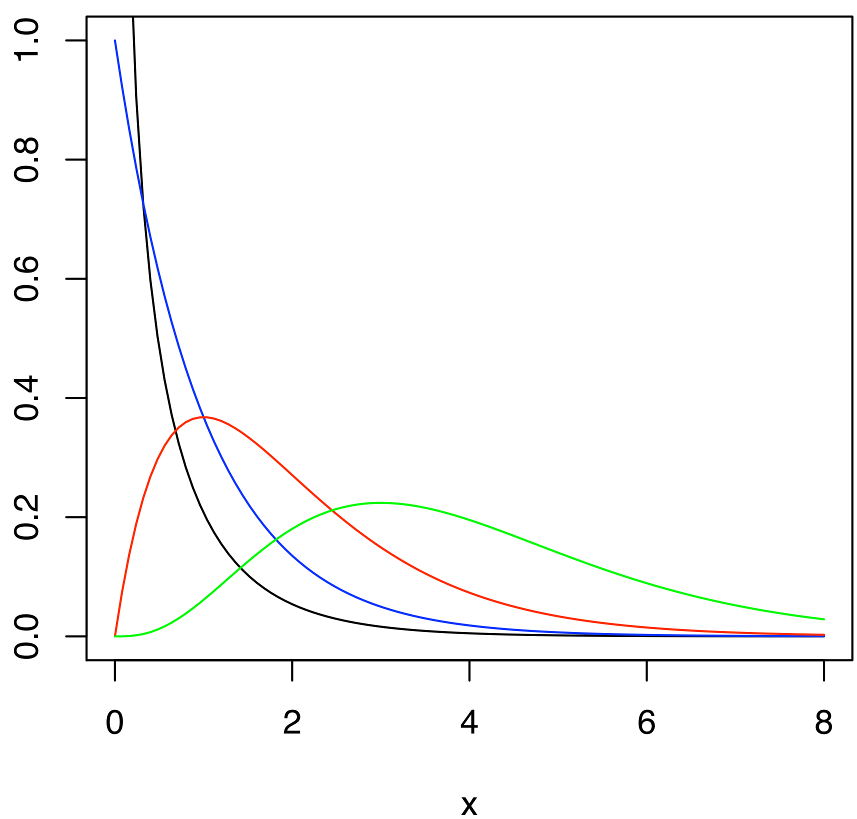

Figure 1: Density for a gamma random variable. Here, $\beta = 1$, $\alpha = 1/2$ (black), 1(blue), 2(red) and 4 (green)

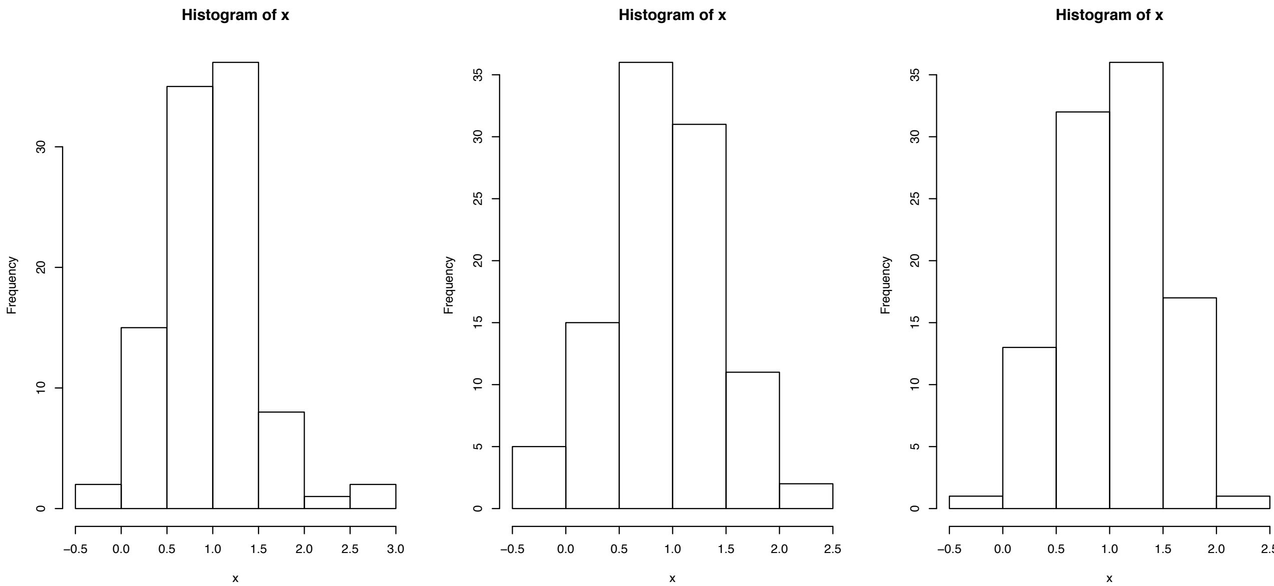

Figure 2: Histrogram of three simulations of 200 normal random variables, mean 1, standard deviation 1/2

```
> curve(dgamma(x,.5,1),0,8,ylim=c(0,1))
> curve(dgamma(x,1,1),0,8,add=TRUE,col="blue")
> curve(dgamma(x,2,1),0,8,add=TRUE,col="red")
> curve(dgamma(x,4,1),0,8,add=TRUE,col="green")
```

4. (beta) $Beta(\alpha, \beta)$ on $[0, 1]$,

$$f_X(x|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}.$$

5. (normal) $N(\mu, \sigma)$ on $\mathbb{R}$,

$$f_X(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

If Z is a standard normal random variable, then $X = \sigma Z + \mu$ has a $N(\mu, \sigma)$ distribution. To simulate 200 normal random variables with mean 1 and standard deviation 1/2, use the R command `x<- rnorm(200,1.0.5)` Histograms of three simulations are given in the figure below.

6. (chi-squared) χ^2_a on $[0, \infty)$

$$f_X(x|a) = \frac{x^{a/2-1}}{2^{a/2}\Gamma(a/2)} e^{-x/2}.$$

The value a is called the number of **degrees of freedom**. If a is a positive integer, let $Z_1, Z_2, \ldots, Z_a$ be independent standard normal random variables. Then,

$$Z_1^2 + Z_2^2 + \cdots + Z_a^2$$

has a χ^2_a distribution.

7. (Student's t) $t_a(\mu, \sigma^2)$ on $\mathbb{R}$,

$$f_X(x|a) = \frac{\Gamma((a+1)/2)}{\sqrt{\alpha\pi}\Gamma(\alpha/2)\sigma}\left(1 + \frac{(x-\mu)^2}{a\sigma^2}\right)^{-(a+1)/2}.$$

The value a is also called the number of **degrees of freedom**. If $\bar{Z}$ is the mean of n standard normal random variables and S^2 is the sample variance with divison by $n-1$.

$$T = \frac{\bar{Z}}{S/\sqrt{n}}.$$

has a t distribution with $n-1$ degrees of freedom.

8. (Fisher's F) $F_{q,a}$ on $[0, \infty)$,

$$f_X(x|q,a) = \frac{\Gamma((q+a)/2)q^{q/2}a^{a/2}}{\Gamma(q/2)\Gamma(a/2)}x^{q/2-1}(a+qx)^{-(q+a)/2}.$$

The F distribution will make an appearance when we see the analysis of variance test. It arises as the ratio of independent chi-square random variables.

R can compute a variety of values for these standard family of distributions

- `dDIST(x, parameters)` is the probability density or mass function of `DIST` evaluated at x.
- `qDIST(p, parameters)` returns x satisfying $P\{X \le x\} = p$, the p**-th quantile** where X has the given distribution,
- `pDIST(x, parameters)` returns $P\{X \le x\}$ where X has the given distribution.
- `rDIST(n, parameters)` generates n random variables having the given distribution.

Exercise 2. *The skewness of a* $\Gamma(\alpha, \beta)$ *random variable is* $2/\sqrt{\alpha}$

Table of Discrete Random Variables

random variable	parameters	mean	variance	generating function
Bernoulli	p	p	$p(1-p)$	$(1-p)+pz$
binomial	n, p	np	$np(1-p)$	$((1-p)+pz)^n$
geometric	p	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)z}$
negative binomial	a, p	$a\frac{1-p}{p}$	$a\frac{1-p}{p^2}$	$\left(\frac{p}{1-(1-p)z}\right)^a$
Poisson	λ	λ	λ	$\exp(-\lambda(1-z))$
uniform	a, b	$\frac{b-a+1}{2}$	$\frac{(b-a+1)^2-1}{12}$	$\frac{z^a}{b-a+1}\frac{1-z^{b-a+1}}{1-z}$

Table of Continuous Random Variables

random variable	parameters	mean	variance	characteristic function
beta	α, β	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$F_{1,1}(a, b; \frac{i\theta}{2\pi})$
chi-squared	a	a	$2a$	$\frac{1}{(1-2i\theta)^{a/2}}$
exponential	λ	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{i\lambda}{\theta+i\lambda}$
F	q, a	$\frac{a}{a-2}, a > 2$	$2a^2\frac{q+a-2}{q(a-4)(a-2)^2}$	
gamma	α, β	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(\frac{i\beta}{\theta+i\beta}\right)^\alpha$
normal	μ, σ^2	μ	σ^2	$\exp(i\mu\theta - \frac{1}{2}\sigma^2\theta^2)$
t	a, μ, σ^2	$\mu, a > 1$	$\sigma^2\frac{a}{a-2}, a > 1$	
uniform	a, b	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$-i\frac{\exp(i\theta b)-\exp(i\theta a)}{\theta(b-a)}$