Moments and Generating Functions

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Some choices of g yield a specific name for the value of $Eg(X)$.

1 Moments, Factorial Moments, and Central Moments

- For $g(x) = x$, we call EX the **mean** of X and often write μ_X or simply μ if only the random variable X is under consideration.
  - S, the number of successes in n Bernoulli trials with success parameter p, has mean np.
  - The mean of a geometric random variable with parameter p is $1/p$.
  - The mean of a exponential random variable with parameter β is β.
  - A standard normal random variable has mean 0.
- For $g(x) = x^m$, EX^m is called the **m-th moment** of X.
  - If X is a Bernoulli random variable, then $X = X^m$. Thus, $EX^m = EX = p$.
  - For a uniform random variable on $[0, 1]$, the m-th moment is $\int_0^1 x^m\,dx = 1/(m + 1)$.
  - The third moment for Z, a standard normal random, is 0. The fourth moment,

$$\begin{aligned} EZ^4 &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^4 \exp\left(-\frac{z^2}{2}\right)dz = -\frac{1}{\sqrt{2\pi}}\left(z^3\exp\left(-\frac{z^2}{2}\right)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 3z^2\exp\left(-\frac{z^2}{2}\right)dz\right) \\ &= 3EZ^2 = 3 \end{aligned}$$

$$\begin{array}{ll} u(z) = z^3 & v(t) = -\exp\left(-\frac{z^2}{2}\right) \\ u'(z) = 3z^2 & v'(t) = z\exp\left(-\frac{z^2}{2}\right) \end{array}$$

  - For T, an exponential random variable, we integrate by parts

$$\begin{aligned} ET^m &= \int_0^{\infty} t^m \frac{1}{\beta}\exp-(t/\beta)\,dt = t^m\exp-(t/\beta)\Big|_0^{\infty} + \int_0^{\infty} mt^{m-1}\exp-(t/\beta)\,dt \\ &= \beta m\int_0^{\infty} t^{m-1}\frac{1}{\beta}\exp-(t/\beta)\,dt = m\beta ET^{m-1} \end{aligned}$$

$$\begin{array}{ll} u(t) = t^m & v(t) = \exp-(t/\beta) \\ u'(t) = mt^{m-1} & v'(t) = \frac{1}{\beta}\exp-(t/\beta) \end{array}$$

    Thus, by induction, we have that

$$ET^m = \beta^m m!.$$

- If $g(x) = (x)_k$, where $(x)_k = x(x-1)\cdots(x-k+1)$, then $E(X)_k$ is called the k-**th factorial moment.**
  - If X is uniformly distributed on $\{1, 2, \ldots, n\}$, then

$$E(X)_k = \sum_{x=1}^{n} (x)_k \frac{1}{n} = \frac{1}{n} \cdot \frac{1}{k+1} \sum_{x=1}^{n} \Delta_+ (x)_{k+1} = \frac{1}{n} \cdot \frac{(n+1)_{k+1}}{k+1}$$

- If $g(x) = (x-\mu)^k$, then $\mu_k = E(X-\mu)^k$ is called the k-**th central moment.**
- The second central moment $\sigma_X^2 = E(X-\mu)^2$ is called the **variance.** Note that

$$\mathrm{Var}(X) = E(X-\mu)^2 = EX^2 - 2\mu EX + \mu^2 = EX^2 - 2\mu^2 + \mu^2 = EX^2 - \mu^2.$$

  σ_X, the square root of the variance, is called the **standard deviation**.
  The **standardized version** of X is

$$Z = \frac{X - \mu_X}{\sigma_X}.$$

  It has mean 0 and variance 1. It is also called z-**value**, z-**score**, and **normal scores**. Note that

$$\mathrm{Var}(aX+b) = E[(aX+b) - (a\mu_X + b))^2] = E[(aX - a\mu_X)^2] = a^2 E[(X-\mu_X)^2] = a^2 \mathrm{Var}(X)$$

  and the standard deviation

$$\sigma_{aX+b} = |a|\sigma_X.$$

  - If X is a Bernoulli random variable, the variance

$$\mathrm{Var}(X) = EX^2 - \mu_X^2 = p - p^2 = p(1-p).$$

  - If U is uniform random variable on $[0, 1]$, then

$$\mathrm{Var}(U) = EU^2 - \mu_U^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

  - If X is uniformly distributed on $\{1, 2, \ldots, n\}$, then

$$\begin{aligned}
\mathrm{Var}(X) &= EX^2 - \mu_X^2 = EX(X-1) + \mu_X - \mu_X^2 \\
&= \frac{1}{n} \cdot \frac{(n+1)n(n-1)}{3} + \frac{n+1}{2} - \frac{(n+1)^2}{4} \\
&= (n+1)\left(\frac{(n-1)}{3} + \frac{1}{2} - \frac{n+1}{4}\right) \\
&= \frac{n+1}{12}(4(n-1) + 6 - 3(n+1)) = \frac{n+1}{12}(n-1) = \frac{1}{12}(n^2-1).
\end{aligned}$$

  - If Z is a standard normal random , then $EZ = 0$, thus $\mathrm{Var}(Z) = EZ^2 = 1$.
  - For T, an exponential random variable,

$$\mathrm{Var}(T) = ET^2 - (ET)^2 = 2\beta^2 - \beta^2 = \beta^2.$$

– The **skewness** of a random variable X

$$\alpha_3 = \frac{\mu_3}{\sigma^3},$$

is the third moment of the standardized version of X.

– The **kurtosis** of a random variable X compares the fourth moment of the standardized version of X to that of a standard normal random variable.

$$\alpha_4 = \frac{\mu_4}{\sigma^4} - 3.$$

2 Generating Functions

For generating functions, it is useful to recall that if h has a converging infinite Taylor series in a interval about the point $x = a$, then

$$h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(a)}{n!}(x-a)^n$$

Where $h^{(n)}(a)$ is the n-th derivative of h evaluated at $x = a$.

- If $g(x) = \exp(i\theta x)$, then

  $$\phi_X(\theta) = E\exp(i\theta X)$$

  is called the **Fourier transform** or the **characteristic function**. Because $|g(x)| = |\exp(i\theta x)| = 1$, the expectation exists for any random variable.

- Similarly, $g(x) = \exp(tx)$, then

  $$M_X(t) = E\exp(tX)$$

  is called the **Laplace transform** or the **moment generating function**. To see the basis for this name, note that if we can reverse the order of differentiation and integration, then

  $$\begin{array}{ll} \frac{d}{dt}M_X(t) = EX\exp(tX) & \frac{d}{dt}M(0) = EX \\ \frac{d^2}{dt^2}M_X(t) = EX^2\exp(tX) & \frac{d^2}{dt^2}M(0) = EX^2 \\ \vdots & \vdots \\ \frac{d^k}{dt^k}M_X(t) = EX^k\exp(tX) & \frac{d^k}{dt^k}M(0) = EX^k \end{array}$$

  – For a standard normal random variable Z,

  $$\begin{aligned} M_Z(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx}\exp\left(-\frac{z^2}{2}\right)dz = e^{t^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{z^2-2tx+t^2}{2}\right)dz \\ &= e^{t^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{(z-t)^2}{2}\right)dz = e^{t^2/2}. \end{aligned}$$

  Thus,

  $$M_Z(t) = \sum_{k=0}^{\infty}\frac{d^k}{dt^k}M(0)\frac{t^k}{k!} = \sum_{n=0}^{\infty}\frac{t^{2n}}{n!2^n} = e^{t^2/2}.$$

  Thus, the odd moments of the standard normal is zero.

  $$EZ^{2n} = \frac{(2n)!}{n!2^n}.$$

– For an exponential random variable T and for $t < 1/\beta$,

$$\begin{aligned} M_T(t) &= \int_0^\infty e^{tu}\frac{1}{\beta}\exp -(u/\beta)\,du = \int_0^\infty \frac{1}{\beta}\exp u(t-1/\beta)\,du \\ &= \frac{1}{1-\beta t} = \sum_{k=0}^\infty (\beta t)^k \end{aligned}$$

and the moments of T can be determined by examining the power series.

- The **cumulant generating function** is defined to be

$$K_X(t) = \log M_X(t).$$

The k-th terms in the Taylor series expansion at 0,

$$k_n(X) = \frac{1}{k!}\frac{d^k}{dt^k}K_X(0)$$

is called the **k-th cumulant**. For the first and second cumulant,

$$\begin{aligned} K_X'(t) &= \tfrac{M_X'(t)}{M_X(t)} & K_X'(0) &= M_X'(0) = EX \\ K_X''(t) &= \tfrac{M_X(t)M_X''(t)-M_X'(t)^2}{M_X(t)^2} & K_X''(0) &= M_X''(0) - M_X'(0)^2 = EX^2 - (EX)^2 = \mathrm{Var}(X) \end{aligned}$$

- If X is $\mathbb{Z}^+$-valued and $g(x) = z^x$, then

$$\rho_X(z) = Ez^X = \sum_{x=0}^\infty P\{X = x\}z^x = \sum_{x=0}^\infty f_X(x)z^x$$

is called the **(probability) generating function**. In the complex variable z, ρ is an analytic function. Its radius of convergence is at least 1 and thus we can obtain the derivatives of ρ by differentiating term by term. At $z = 0$ we obtain,

$$f_X(x) = \frac{1}{x!}\frac{d^x}{dz^x}\rho_X(0).$$

If we can take the derivative at $z = 1$, we have

$$\begin{aligned} \tfrac{d}{dz}\rho_X(z) &= EXz^{X-1} & \rho(1) &= EX \\ \tfrac{d^2}{dz^2}\rho_X(z) &= EX(X-1)z^{X-2} & \rho(1) &= E(X)_2 \\ &\vdots & &\vdots \\ \tfrac{d^k}{dz^k}\rho_X(z) &= E(X)_k z^{X-k} & \rho(1) &= E(X)_k \end{aligned}$$

– For X, a geometric random variable,

$$\rho_X(z) = Ez^X = \sum_{x=0}^\infty f_X(x)z^x = \sum_{x=1}^\infty p(1-p)^{x-1}z^x = \frac{pz}{1-(1-p)z}.$$

$$\rho'_X(z) = \frac{p}{(1-(1-p)z)^2} \qquad \rho_X(1) = \frac{1}{p}$$
$$\rho''_X(z) = \frac{2p(1-p)}{(1-(1-p)z)^3} \qquad \rho_X(2) = \frac{2(1-p)}{p^2}$$

and

$$\begin{aligned} \mathrm{Var}(X) &= EX^2 - (EX)^2 = EX(X-1) + EX - (EX)^2 \\ &= \frac{2(1-p)}{p^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{2(1-p)+p-1}{p^2} = \frac{1-p}{p^2} \end{aligned}$$

– For X, a binomial random variable based on n Bernoulli trials,

$$\rho_X(z) = Ez^X = \sum_{x=0}^{n} f_X(x)z^x = \sum_{x=0}^{n} \binom{n}{x} p^x(1-p)^{n-x}z^x = ((1-p)+zp)^n.$$

by the binomial theorem.

$$\rho''_X(z) = n(n-1)p^2((1-p)+zp)^{n-2} \quad EX(X-1) = \rho''_X(1) = n(n-1)p$$

and

$$\begin{aligned} \mathrm{Var}(X) &= EX^2 - (EX)^2 = EX(X-1) + EX - (EX)^2 \\ &= n(n-1)p^2 + np - (np)^2 = np - np^2 = np(1-p). \end{aligned}$$

The moment generating function $M_X(t) = \rho_X(e^t)$.