

Joint Distributions and Inequalities

Homework 8

Problems

1. Roll two fair die. Let X_1 be the larger of the two values and X_2 be the smaller.
 - (a) Give the joint mass function for X_1 and X_2 .
 - (b) Find $P\{X_1 = X_2\}$.
 - (c) Give the marginal mass functions.
 - (d) Find $E[X_2|X_1]$.
2. Let X_1 and X_2 be jointly continuous random variables with density

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} x_2 \exp(-x_1 x_2), & x_1 \geq 0, 0 \leq x_2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Verify that this is a bona fide density function.
- (b) Find the two marginal densities.
- (c) Find the two conditional densities.
- (d) Find $P\{X_1 > 1|X_2 < 1/2\}$.
- (e) Find $E[X_2|X_1]$.
- (f) Find $\text{Var}(X_1|X_2)$.

Challenging Problems

1. An urn that contains m white marbles and n black marbles. Draw k with replacement. Let $X_1 = 1$ if the first marble is white, 0 if it is black. Set S to be the total number of white marbles in the k draws.
 - (a) Find the joint mass function for X_1 and S .
 - (b) Find $P\{X_1 = 1|S = s\}$
 - (c) Repeat (a) and (b) drawing without replacement.
2. Let $X \sim \text{Bin}(n, p)$. Thus, $EX = np$
 - (a) Find the cumulant generating $K_X(t)$ function for X
 - (b) Find the value $t^*(x)$ that satisfies $K'_X(t) = x$.
 - (c) Find the rate function $K_X^*(t)$ for X .
 - (d) Find the Chernoff bound for $P\{X > nq\}, q > p$
 - (e) Compare this bound for $p = 1/2$ and $q = 0.51, 0.52, \dots, 0.70$ to the actual value obtained by using the `pbinom` command in R.