Conditional Distributions and Transformations

Homework 10

Problems

1. Prove the law of total covariance.

$$\mathrm{Cov}(Y_1, Y_2) - E[\mathrm{Cov}(Y_1, Y_2|X)] = \mathrm{Cov}(E[Y_1|X]E[Y_2|X]).$$

2. Consider the covariance matrix

$$\Sigma = \begin{pmatrix} 21 & 8 \\ 8 & 9 \end{pmatrix}$$

   (a) Find $\det(\Sigma)$.
   (b) Find the eigenvalues and eigenvectors for Σ.
   (c) Give a linear transformation $Y = AZ$ of two independent standard normal random variables Z_1 and Z_2 so that $\mathrm{Cov}(Y) = \Sigma$.

3. Let X_1, X_2 have density

$$f_{X_1,X_2}(x_1, x_2) = \frac{\Gamma\left(\sum_{i=1}^3 \alpha_i\right)}{\prod_{i=1}^3 \Gamma(\alpha_i)} x_1^{\alpha_1-1} x_2^{\alpha_2-1} (1 - x_1 - x_2)^{\alpha_3-1}, \quad x_1, x_2 \in [0, 1],\ x_1 + x_2 < 1$$

   $\alpha_i > 0, i = 1, 2, 3.$

   (a) Find $f_{X_1}(x_1)$. the density of X_1.
   (b) Find $f_{X_2|X_1}(x_2|x_1)$
   (c) Find $E[X_1 X_2]$
   (d) Find $\mathrm{Cov}(X_1, X_2)$.

Challenging Problems

1. Consider an urn with ℓ white, m gray, anf n black balls. Draw k. Let W be the number of black balls, G be the number of gray balls, and B be the number of black balls.

   (a) Find the mass function $f_{W,B}(w, b)$ for the number of white and black marbles.
   (b) Find EW, EG, EB.
   (c) Find $E[W|G]$ and $E[B|G]$.
   (d) Find $\mathrm{Cov}(W, B|G)$
   (e) Find $\mathrm{Cov}(W, B)$

2. Let Z_1, Z_2 be a bivariate standard normal with correlation ρ. Find the correlation of Z_1^2 and Z_2^2.