Convergence in Distribution& the Central Limit Theorem

Homework 12

Problems

1. Let $X_1, X_2, \ldots, X_n$ are independent geometric random variables with parameter $1/3$.
   (a) Give the mean and standard deviation for these random variables.
   (b) Compute the skewness of these random variables.
   (c) Let $Y_n = (X_1 + \cdots + X_n)/n$. Find the mean, variance, and skewness of Y_n.
   (d) Using Sugden's criterion, find the minimum sample size for the application of the central limit theorem.
   (e) Simulate 10000 observations of Y_n for $n = 64$. Make histograms and describe them.
   (f) Estimate $P\{Y_{64} < 2.3\}$ using the central limit theorem and compare this to the value given by the simulation.
   (g) Estimate the value of y so that $P\{Y_{64} > y\} = 0.90$ using both the central limit theorem, the nagative binomial distribution, and the `quantile` command.

2. For small angles, the period of a pendulum is

$$T = 2\pi\sqrt{\frac{L}{g}}. \tag{1}$$

   Here L is the length of pendulum and g is the acceleration of gravity.

   In 2005, the Huygens space probe landed on the surface of Titan. Pictures of Titan were taken with a camera designed at the University of Arizona. We will use this relationship and observations from a swinging pendulum to estimate the acceleration of gravity on the moon.

   (a) Give an expression for g in terms of the other quantities in equation (1).
   (b) Assume that the length $L = 1$ meter. Now make repeated independent measurements, $T_1, T_2, \cdots T_{25}$, of the period. If these measurements have mean 5.40 seconds and standard deviation 0.12 seconds, give the mean and standard deviation of $\bar{T}$.
   (c) Use the expression in part (a) to create an estimator $\hat{g}$ for g. Using the delta method, estimate the mean and standard deviation of $\hat{g}$.
   (d) Simulate this process 1000 times using the `rnorm` command and compare the standard deviation of the estimates to the value given by the delta method.

Challenging Problems

1. For the delta method example on bird fecunity, minimize over the choices for n_F, n_p, n_N

$$\left(\frac{\sigma_{\hat{B}}}{B}\right)^2 = \frac{1}{n_F}\left(\frac{\sigma_F}{F}\right)^2 + \frac{1}{n_p}\left(\frac{1-p}{p}\right) + \frac{1}{n_N}\left(\frac{\sigma_N}{N}\right)^2.$$

   subject to a fixed cost $C = c_F n_F + c_p n_p + c_N n_N$ where c_F, c_p, c_N are the cost for each observation from each source. (This can be determined usng Lagrange multipliers.)

2. Show that the t_ν density converges to the $N(0,1)$ as the number of degrees of freedom $\nu \to \infty$. (Use Stirling's formula.)