

# Joint Distributions and Inequalities

## Homework 8

### Problems

1. Roll two fair die. Let  $X_1$  be the larger of the two values and  $X_2$  be the smaller.
  - (a) Give the joint mass function for  $X_1$  and  $X_2$ .
  - (b) Find  $P\{X_1 = X_2\}$ .
  - (c) Give the marginal mass functions.
  - (d) Find  $E[X_2|X_1]$ .
2. Let  $X_1$  and  $X_2$  be jointly continuous random variables with density

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} x_2 \exp(-x_1 x_2), & x_1 \geq 0, 0 \leq x_2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Verify that this is a bona fide density function.
- (b) Find the two marginal densities.
- (c) Find the two conditional densities.
- (d) Find  $P\{X_1 > 1|X_2 < 1/2\}$ .
- (e) Find  $E[X_2|X_1]$ .
- (f) Find  $\text{Var}(X_1|X_2)$ .

### Challenging Problems

1. An urn that contains  $m$  white marbles and  $n$  black marbles. Draw  $k$  with replacement. Let  $X_1 = 1$  if the first marble is white, 0 if it is black. Set  $S$  to be the total number of white marbles in the  $k$  draws.
  - (a) Find the joint mass function for  $X_1$  and  $S$ .
  - (b) Find  $P\{X_1 = 1|S = s\}$
  - (c) Repeat (a) and (b) drawing without replacement.
2. Let  $X \sim \text{Bin}(n, p)$ . Thus,  $EX = np$ 
  - (a) Find the cumulant generating  $K_X(t)$  function for  $X$
  - (b) Find the value  $t^*(x)$  that satisfies  $K'_X(t) = x$ .
  - (c) Find the rate function  $K_X^*(t)$  for  $X$ .
  - (d) Find the Chernoff bound for  $P\{X > nq\}, q > p$
  - (e) Compare this bound for  $p = 1/2, n = 100$ , and  $q = 0.51, 0.52, \dots, 0.70$  to the actual value obtained by using the `pbinom` command in R.