Law of Large Numbers & Convergence in Distribution

Homework 11

Problems

1. Consider the function $g(x) = 1 + \sin(x^2)$

   (a) Plot g on the interval from 0 to 4.

   (b) Estimate the integral $\int_0^4 g(x)\,dx$ using Monte Carlo simulations with 100 uniform random variables on the interval $[0, 5]$.

   (c) Repeat the estimate 400 times and use this to estimate the standard deviation of the error in the integral.

   (d) Find the mean of these 400 simulations. This is a single Monte Carlo simulation based on 40,000 uniform random variables. Use part (c) to give an estimate of the standard deviation of the simulation based on 40,000 uniform random variables

   (e) Use the `integrate` command to compute numerically the integral.

   (f) Using your estimate for the standard deviation, discuss how close your estimate is to the numerical value. In particular, give the number of standard deviations your Monte Carlo values are from the numerical evaluation of the integral in part (e)

2. Consider the function $g(x) = \sin(x^2)/x^{5/2}$.

   (a) Give a plot of g on the interval $(0, 4]$.

   (b) Use simple Monte Carlo simulation to provide 100 estimates of $\int_0^4 g(x)\,dx$ based on a sample of 200 uniform random variables. Find the mean and standard deviation of these simulations.

   (c) Check that

   $$f_X(x) = \begin{cases} \frac{1}{4\sqrt{x}}, & 0 \le x \le 4, \\ 0 & \text{otherwise} \end{cases}$$

   is a valid density function.

   (d) Find the distribution function and the probability transform.

   (e) What is the weight function?

   (f) Use importance sampling to provide 100 estimates of $\int_0^4 g(x)\,dx$ based on a sample of 200 random variables with proposal density f_X. Find the mean and standard deviation of these simulations.

   (g) Find the ratio of the standard deviations and the variances of these two estimates. Which method does better?

Challenging Problems

1. (a) Assume that $c_k \to 0$ and $a_k \to \infty$ so that $a_k c_k \to \lambda$, show that $(1 + c_k)^{a_k} \to \exp \lambda$

   (b) (birthday problem) Let $X_1, X_2, \dots$ be independent and uniform on $\{1, \dots, N\}$. Let

$$T_N = \min\{n : X_n = X_m \text{ for some } m < n\}.$$

   Then

$$P\{T_N > n\} = \prod_{m=2}^{n} \left(1 - \frac{m-1}{N}\right).$$

   Show that

$$\lim_{N \to \infty} P\{\frac{T_N}{\sqrt{N}} > x\} = \exp\left(-\frac{x^2}{2}\right).$$

   (c) For the case $N = 365$,

$$P\{T_N > n\} \approx \exp\left(-\frac{n^2}{730}\right).$$

   For the choice $n = 22$, find both the exact probability and its approximation.

2. Let $X_p \sim Negbin(n, p)$. Find the limiting distribution of pX_p as $p \to 0$.