Convergence in Distribution& the Central Limit Theorem

Homework 12

Problems

1. Let $X_1, X_2, \dots, X_n$ are independent geometric random variables with parameter $1/4$.
   (a) Give the mean and standard deviation for these random variables.
   (b) Compute the skewness of these random variables.
   (c) Let $Y_n = (X_1 + \cdots + X_n)/n$. Find the mean, variance, and skewness of Y_n.
   (d) Using Sugden's criterion, find the minimum sample size for the application of the central limit theorem.
   (e) Simulate 10000 observations of Y_n for $n = 100$. Make histograms and describe them.
   (f) Estimate $P\{Y_{100} > 3\}$ using the central limit theorem and compare this to the value given by the simulation.
   (g) Estimate the value of y so that $P\{Y_{100} < y\} = 0.05$ using both the central limit theorem, the nagative binomial distribution, and the `quantile` command.

2. (Buffon's needle problem) "Suppose we have a floor made of parallel strips of wood, each the same width ℓ, and we drop a needle, length ℓ, onto the floor. What is the probability that the needle will lie across a line between two strips?"

   The answer, $1/\pi$, gives us a way to estimate π. Just drop the needle n times and find the mean number of times that the needle lies across a line. For the i-th needle, set $B_i = 1$ if the needle crosses a line and 0 if it does not. Consequently,

$$EB_i = \frac{1}{\pi}, \quad \mathrm{Var}(B_i) = \frac{\pi - 1}{\pi^2}.$$

   Drop n needles. The case $n = 10$ is shown in the figure.
   (a) Find $E\bar{B}$.
   (b) $\mathrm{Var}(\bar{B})$.
   (c) Estimate π by using $\hat{\pi} = 1/\bar{B}$. Use the delta method to estimate the standard deviation $\sigma_{\hat{\pi}}$ for $n = 1600$. (Hint: Use $g(b) = 1/b$ and evaluate at $b = 1/\pi$.)

Challenging Problems

1. For the delta method example on bird fecunity, minimize over the choices for n_F, n_p, n_N

$$\left(\frac{\sigma_{\hat{B}}}{B}\right)^2 = \frac{1}{n_F}\left(\frac{\sigma_F}{F}\right)^2 + \frac{1}{n_p}\left(\frac{1-p}{p}\right) + \frac{1}{n_N}\left(\frac{\sigma_N}{N}\right)^2.$$

   subject to a fixed cost $C = c_F n_F + c_p n_p + c_N n_N$ where c_F, c_p, c_N are the cost for each observation from each source. (This can be determined usng Lagrange multipliers.)

2. Show that the t_ν density converges to the $N(0,1)$ as the number of degrees of freedom $\nu \to \infty$. (Use Stirling's formula.)