Convergence in Distribution& the Central Limit Theorem

Homework 13

Problems

1. The focal length f of an optical instrument is needed. This is determined by using the thin lens formula,

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{f}.$$

where r_1 is the distance from the lens to the object and r_2 is the distance from the lens to the real image of the object. The distance r_1 is independently measured 12 times and r_2 is independently measured 40 times. The mean of the measurements is the actual distances, 9 centimeters and 18 centimeters, respectively. The standard deviation of the measurement is 0.1 centimeters for r_1 and 0.5 centimeter for r_2.

(a) Let $\bar{R}_1$ be the sample mean of the 12 measurements to the object. Estimate, using the central limit theorem, $P\{\bar{R}_1 < 8.9\text{cm}\}$.

(b) Let $\bar{R}_2$ be the sample mean of the 40 measurements to the image. Estimate, using the central limit theorem, $P\{\bar{R}_2 < 17.9\text{cm}\}$.

(c) How many measurements are needed so that $P\{|\bar{R}_2 - 18\text{cm}| > 0.1\text{cm}\} \leq 0.02$.

(d) Would more or fewer measurements be needed so that

i. $P\{|\bar{R}_2 - 18\text{cm}| > 0.1\text{cm}\} \leq 0.01$?

ii. $P\{|\bar{R}_2 - 18\text{cm}| > 0.2\text{cm}\} \leq 0.02$?

Explain your answer.

(e) For measurements $r_{1,1}, \ldots, r_{1,12}$ and $r_{2,1}, \ldots, r_{2,40}$, estimate the focal length using

$$\frac{1}{\bar{r}_1} + \frac{1}{\bar{r}_2} = \frac{1}{\hat{f}}.$$

Use the delta method to give an estimate of the standard deviation of $\hat{f}$.

(f) Simulate this protocol 10000 times using the `rnorm` command and compare the results from your simulation to the results from the delta method.

Challenging Problems

1. Let $X_1, X_2, \ldots,$ be independent and identically distributed non-negative continuous random variables with density $f_X(x)$ that is continuous on $[0, \infty)$. Define

$$M_n = \min\{X_1, X_2, \ldots, X_n\}.$$

Show that nM_n converges in distribution to a $Exp(f_X(0+))$ random varaible.

2. Let Let $X_1, X_2, \ldots,$ be independent and identically distributed discrete random variable with mass function

x	0	1	2
$f_X(x)$	1/4	1/2	1/4

(a) Let $S_{30} = X_1 + X_2 \cdots + X_{30}$.Use the central limit theorem along with a continuity correciiion to estimate the probability that $P_{\{}S_{30} > 33\}$.

(b) Use 100,000 simulations to estimate this probability.