

Overview

Homework 1

Problems

1. Consider a Bayesian approach to Bernoulli trials. Let the prior density be $Beta(4, 5)$ for the parameter $\theta \in [0, 1]$ and assume that out of 30 trials, we have $x = 16$ successes.
 - (a) Describe the posterior density $f_{\Theta|X}(\theta|16)$ for the probability of heads.
 - (b) Give the posterior mean $E[\Theta|X = 16]$ for the probability of heads.
 - (c) Find values ℓ and u so that $P\{\Theta < \ell|X = 16\} = P\{\Theta > u|X = 16\} = 0.025$. The interval (ℓ, u) is called the 95% equal-tailed credible interval.
2. Let $X_1, \dots, X_{20} \sim N(\mu, 1)$. Let Φ denote the cumulative distribution function for the standard normal. For $\alpha \in (0, 1)$, define z_α so that $1 - \Phi(z_\alpha) = \alpha$.
 - (a) Give an expression for $\beta(\mu) = P_\mu \left\{ \left| \frac{\bar{X}}{1/\sqrt{20}} \right| > z_\alpha \right\}$ in terms of Φ .
 - (b) Provide a plot of $\beta(\mu)$ versus μ for $\mu \in [-1, 1]$ for both $\alpha = 0.01$ and 0.05 . These are power functions for a two-sided test $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$ with significance level α .
3. For a $N(\mu, \sigma^2)$ density,
 - (a) Write the density in the form using the natural parameters for an exponential family.

$$f_{X|H}(x|\eta) = h(x) \exp(\eta_1 t_1(x) + \eta_2 t_2(x) - A(\eta)).$$

- (b) Find $Et_1(X)$, $Et_2(X)$, $\text{Var}(t_1(X))$, $\text{Var}(t_2(X))$ and $\text{Cov}(t_1(X), t_2(X))$.

Challenging Problems

1. Define (X_1, \dots, X_n) be a multinomial distribution based on m trials with the probability that a trial chooses category $i = 1, \dots, n$ equal to p_i .
 - (a) Determine the distribution of X_n .
 - (b) Give the parameters in a multinomial distribution for conditional mass function (X_1, \dots, X_{n-1}) given that $X_n = x_n$.
 - (c) Describe how a multinomial distribution can be considered as a hierarchical model of binomial distributions.
2. For n independent $Pois(\lambda)$ random variables $T(X) = X_1 + \dots + X_n$ is a sufficient statistic.

- (a) Determine the distribution of conditional mass function

$$P\{X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4 | T(X) = t\}$$

- (b) Give 10 simulations of X_1, X_2, X_3, X_4 when $T(X) = 17$. (The `rmultinom` command will be useful here.)