Sufficiency & Bias

Homework 3

Problems

1. Let $V_n \sim \chi^2_n$
   (a) Find
   $$E\left[\frac{1}{V_n}\right].$$
   (b) Find the density for $1/V_n$, the inverse χ^2_n.
2. For capture-recapture, consider the case with population N, t tagged, and k recaptured.
   (a) Use the hypergeometric distribution and the delta method to estimate the variance of $\hat{N}$.
   (b) Apply this to the case where population $N = 4500$, $t = 400$, and $k = 500$.
   (c) Perform 2000 simulations of this experiment and compute the simulation standard deviation.
   (d) Compare this to the answer obtained using the delta method.
3. For Bernoulli trials, $\hat{p}$ is both an unbiased and a maximum likelihood estimate of p, the success probability.
   (a) Give the maximum likelihood estimate for the $p(1-p)$, the variance.
   (b) Find the expected value for this estimate.
   (c) Use the answer to part (b) to find an unbiased estimator for $p(1-p)$.

Challenging Problems

1. For $Z_1, Z_2, \ldots, Z_n$ independent standard normal variables, then
   $$V_{n-1} = \sum_{i=1}^{n} (Z_i - \bar{Z})^2$$
   is χ^2_{n-1}. The sample variance
   $$S_n^2 = \frac{1}{n-1} V_{n-1}$$
   is an unbiased estimator for the variance.
   (a) S_n is a biased estimator for the standard deviation. Give the direction of the bias and explain why.

(b) Show that $ES_n = \sqrt{\frac{2}{n-1}}\ \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}$.

(c) Use 1000 simulation to estimate both ES_n and ES_n^2 for values n from 5 to 30.

(d) Use a plot to compare the values of ES_n to the simulation values.

(e) The Stirling approximation states that

$$\Gamma(t) \approx \sqrt{2\pi(t-1)}\left(\frac{t-1}{e}\right)^{t-1}, \quad t > 1.$$

Show that

$$\lim_{n\to\infty} ES_n = 1.$$

2. Assume that X, the value an unfair die, has the following probability mass function.

x	1	2	3	4	5	6
$f_X(x\|\alpha)$	$\frac{1}{6} - \frac{3\alpha}{6}$	$\frac{1}{6} - \frac{2\alpha}{6}$	$\frac{1}{6} - \frac{\alpha}{6}$	$\frac{1}{6} + \frac{\alpha}{6}$	$\frac{1}{6} + \frac{2\alpha}{6}$	$\frac{1}{6} + \frac{3\alpha}{6}$

(a) Give the possible values for the parameter α.

(b) Find the mean and variance of X.

(c) Find a method of moments estimator $\hat{\alpha}$ for α based on n rolls of the die.

(d) Use the delta to give an estimate of the standard deviation for $\hat{\alpha}$.

(e) Is the estimator biased? Explain your answer.