Composite Hypotheses

Homework 6

Problems

1. Let $X_1, \ldots, X_m \sim N(0, \sigma^2)$ and consider the test

$$H_0 : \sigma = \sigma_0 \quad \text{versus} \quad H_1 : \sigma = \sigma_1,$$

   with $\sigma_0 < \sigma_1$.

   (a) Find a sufficient statistic $T(\mathbf{X})$ for this test.

   (b) Describe how to determine the critical value k_α for an α-level test. Here the critical region

$$\{\mathbf{x}; T(\mathbf{x}) > k_\alpha\}.$$

   (c) Give this value in the case $m = 10$ and $\alpha = 0.05$.

2. Verify that test based on the distributions below have a monotone likelihood ratio. Give the test statistic for this situation.

   (a) $X_1, \ldots, X_m \sim Ber(p)$ with a test on the parameter p

   (b) $X \sim Hyper(t, N - t, k)$ with a test on the parameter N, the total population size. Use r the value for X as the test statistic and the critical region is $\{r \geq r_0\}$. Because we have a "less than" alternative, the ratio is decreasing in the test statistic,

   (c) $X_1, \ldots, X_m \sim Unif(0, \theta)$ with a test on the parameter θ.

3. For a test of proportions

$$H_0 : p \leq p_0 \quad \text{versus} \quad H_1 : p > p_0.$$

   Use the test statistic

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}$$

   and assume that it has a standard normal distribution.

   (a) For $n = 325$, $p_0 = 0.7$, and $\alpha = 0.02$, give the critical value of $\hat{p}$

   (b) Give the power as a function of Φ, the cumulative distribution function of the standard normal, and provide a plot of the power.

Challenging Problems

1. Let $X_1, \ldots, X_m$ be indenpdent from an exponential family using the natural parameter

(a) Describe the test statistic for the simple hypothesis

$$H_0 : \eta = \eta_0 \quad \text{versus} \quad H_1 : \eta = \eta_1,$$

with $\eta_0 > \eta_1$.

(b) Show that this test statistics has a monotone likelihood ratio and thus it is uniformly most powerful for the one-sided test

$$H_0 : \eta \leq \eta_0 \quad \text{versus} \quad H_1 : \eta > \eta_0,$$

2. During meiosis, paired chromosomes experience **crossing over** events in the formation of gametes. Recombination can occur with a small probability at any location along chromosome.Thus, the number of crossing over events can be modeled as Poisson random variable. The mean number of cross overs for a given chromosomal segment is called its **genetic length** with Morgans as the unit of measurement. One simple question is: Are the number of crossing over events different in sperm and in eggs? Using the subscript m for male and f for female, this leads to the hypothesis

$$H_0 : \lambda_m = \lambda_f \quad \text{versus} \quad H_1 : \lambda_m \neq \lambda_f$$

where λ_m and λ_f is the parameter in the Poisson random variable that gives the number of crossing over events in the human chromosome across all 22 autosomes. The data are $X_m = n_m, X_f = n_f$ the number of crossing over events for each parent's chromosome.

(a) Write down the likelihood function $L(\lambda_m, \lambda_f | n_m, n_f)$ for this situation.

(b) Find the maximum for the parameters in the likelihood.

(c) Find the maximum for the allowable parameter values under the null hypothesis.

(d) Find the likelihood ratio.

(e) Under the null hypothesis, $X_m, X_f \sim Pois(\lambda)$. Set $n = n_f + n_m$ and find

$$P\{X_m = n_m | X_m + X_f = n\}.$$

(f) What is this distribution?

(g) Use this as a test statistic on the data $n_m = 56$ and $n_f = 107$ for two individuals sharing the same parents. Find the p-value for this case.