Confidence Intervals

Homework 8

Problems

1. Let $X_1, \ldots, X_n \sim Ber(p)$ be n Bernoulli trials. Hoeffdings inequality for Bernoulli trials states that

$$P_p\{|\hat{p}_n - p| > \epsilon\} \leq 2\exp(-2n\epsilon^2).$$

   Create a two-sided γ-level confidence interval for p using Hoeffdings inequality.

2. Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ where both σ and μ are unknown.

   (a) Let $T(\mathbf{x}) = \sum_{i=1}^n (X_i - \bar{X})^2$. What is the distribution of $T(\mathbf{x})/\sigma^2$.

   (b) Use this to find a γ-level confidence interval for σ^2.

3. Let $X_1, \ldots, X_n \sim Beta(\alpha, 1)$.

   (a) Show that $T(\mathbf{x}) = -\sum_{i=1}^n \ln X_i$ is a sufficient statistic.

   (b) What is the distribution of $T(\mathbf{x})$?

   (c) Find a γ-level confidence interval for α.

Challenging Problems

1. For linear regression, we have $Y_i = \alpha + \beta x_i + \epsilon_i$ where $\epsilon_i \sim N(0, \sigma^2)$, σ^2 is unknown. For data $(x_1, y_1), (x_2, y_2) \ldots, (x_n, y_n)$, the estimator for the slope and intercept are

$$\hat{\beta}(x, y) = \frac{\text{cov}(x, y)}{\text{var}(x)}, \quad \hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}.$$

   Show that

   (a) $\text{Var}_{(\alpha,\beta)}(\bar{Y}) = \frac{\sigma^2}{n}$,

   (b) $\text{Cov}_{(\alpha,\beta)}(\bar{Y}, \hat{\beta}) = 0$

   (c) $\text{Var}_{(\alpha,\beta)}(\hat{\alpha}) = \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{(n-1)\text{var}(x)}\right)$

   (d) $\text{Cov}_{(\alpha,\beta)}(\hat{\alpha}, \hat{\beta}) = -\bar{x}\text{Var}_{(\alpha,\beta)}(\hat{\beta})$

2. Let $(Z_{1,i}, Z_{2,i}), i = 1, \ldots, n$ be independent standard bivariate normal random variables.

   (a) What is the distribution of $\bar{Z}_1^2 + \bar{Z}_2^2$?

(b) Let $(X_{1,i}, X_{2,i}), i = 1, \ldots, n$ be independent bivariate normal random variables with mean vector $\mu = (\mu_1, \mu_2)$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}.$$

Give an γ-level confidence interval for μ,

(c) Describe what this set looks like in the plane.

(d) Repeat parts (b) and (c) with

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$