Central Limit Theorem*

Worksheet 13

1. Let $X_1, X_2, \ldots, X_n$ are independent geometric random variables with parameter 1/3.
   (a) Give the mean and standard deviation for these random variables.
   (b) Let $Y_n = (X_1 + \cdots + X_n)/n$. Find the mean and variance of Y_n.
   (c) Simulate 1000 observations of Y_n for $n = 4, 16$, and 64. Make histograms and describe them.
   (d) Compare the means standard deviation for these simulations and compare them to the answers in part (b).
   (e) The **skewness** of a sample is defined as
   $$\frac{\overline{(y-\bar{y})^3}}{s_y^3}.$$
   Give the skewness for the three simulations in part (c). What do you see?
   (f) Estimate $P\{Y_{64} < 2.3\}$ using the central limit theorem and compare this to the value given by the simulation.
   (g) Estimate the value of y so that $P\{Y_{64} > y\} = 0.90$ using both the central limit theorem and the `quantile` command.

2. For small angles, the period of a pendulum is
   $$T = 2\pi\sqrt{\frac{L}{g}}. \qquad (1)$$
   Here L is the length of pendulum and g is the acceleration of gravity.

   In 2005, the Huygens space probe landed on the surface of Titan. Pictures of Titan were taken with a camera designed at the University of Arizona. We will use this relationship and observations from a swinging pendulum to estimate the acceleration of gravity on the moon.
   (a) Give an expression for g in terms of the other quantities in equation (1).
   (b) Assume that the length $L = 1$ meter. Now make repeated independent measurements, $T_1, T_2, \cdots T_{25}$, of the period. If these measurements have mean 5.40 seconds and standard deviation 0.12 seconds, give the mean and standard deviation of $\bar{T}$.
   (c) Use the expression in part (a) to create an estimator $\hat{g}$ for g. Using the delta method, estimate the mean and standard deviation of $\hat{g}$.
   (d) Simulate this process 1000 times using the `rnorm` command and compare the standard deviation of the estimates to the value given by the delta method.