Linear Algebra Basics

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1 Matrix Algebra Operations

Example 1. *For $i = 1, 2$ and $j = 1, 2, 3$, let*

$$a_{ij}$$

be the quantity store i orders of product j, We can write this conveniently as a **matrix**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

If the price of these objects is x_1, x_2, x_3 we can write this as a **column vector**

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The total bill for each store

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

is simply the **matrix product** *of the matrix A with the vector $\mathbf{x}$ and write $A\mathbf{x}$.*

Let look at this in a more abstract setting.

- Let C_{ij} denote the entry in the i-th row and j-th column of a matrix C.
- Matrices can be multiplied by a scalar a. So, the matrix aC has ij entry

$$(aC)_{ij} = aC_{ij}$$

- If two matrices, A and B have the same number of rows and columns, then we can form their sum $A + B$ with

$$(A + B)_{ij} = A_{ij} + B_{ij}.$$

- A matrix A with r_A rows and c_A and a matrix B with r_B rows and c_B columns can be multiplied to form a matrix AB provide that $c_A = r_B$, the number of columns in A equals the number of rows in B. In this case

$$(AB)_{ij} = \sum_{k=1}^{c_A} A_{ik}B_{kj}.$$

Exercise 2. *Let*

$$A = \begin{pmatrix} 1 & 3 & 3 & -1 \\ 0 & 8 & 12 & 2 \\ 3 & 2 & 0 & 2 \\ 1 & 0 & 0 & 2 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 4 & 6 & 1 & 0 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

Find AB and BA

- Matrix multiplication is **associative**

  $$(AB)C = A(BC)$$

  and distributive

  $$A(B + C) = AB + AC$$

- The **transpose** of a matrix is obtained by reversing the rows and columns of a matrix. We use a superscript T to indicate the transpose. Thus, the ij entry of a matrix C is the ji entry of its transpose, C^T.

  Example 3.

  $$\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 7 \end{pmatrix}^T = \begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 3 & 7 \end{pmatrix}$$

- The d-dimensional **identity matrix** I is the matrix with the value 1 for all entries on the diagonal ($I_{jj} = 1, j = 1 \ldots, d$) and 0 for all other entries.

  Exercise 4. *d-dimensional vector x,*

  $$Ix = x.$$

- A $d \times d$ matrix C is called invertible with **inverse** C^{-1} provided that

  $$CC^{-1} = C^{-1}C = I.$$

  Only one matrix can have this property.

- Suppose we have a d-dimensional vector a of known values and a $d \times d$ matrix C and we want to determine the vectors x that satisfy

  $$b = Cx.$$

  This equation could have no solutions, a single solution, or an infinite number of solutions. If the matrix C is invertible, then we have a single solution

  $$x = C^{-1}b. \quad (C^{-1}b = C^{-1}(Cx) = (C^{-1}C)x = Ix = x)$$

2 Solving Linear Equations

We will perform the task of finding the matrix inverse using **Gaussian elimination** or the **Gauss-Jordan algorithm**. The operations on matrices that are permitted are:

- **Row switching**
  A row within the matrix can be switched with another row.
  $$R_i \leftrightarrow R_j$$
- **Row multiplication**
  Each element in a row can be multiplied by a non-zero constant.
  $$kR_i \to R_i, \text{ where } k \neq 0$$
- **Row addition**
  A row can be replaced by the sum of that row and a multiple of another row.
  $$R_i + kR_j \to R_i, \text{where } i \neq j$$

Example 5. *For*

$$\begin{array}{rrl} x_1 & +5x_2 & = 7 \\ -2x_1 & -7x_2 & = -5 \end{array}$$

we write the **augmented matrix** $(A|x)$.

$$\left(\begin{array}{cc|c} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array}\right)$$

Then $2R_1 + R_2 \to R_2$

$$\left(\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 3 & 9 \end{array}\right)$$

Then $R_2/3$

$$\left(\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array}\right)$$

Then $-5R_2 + R_1 \to R_1$

$$\left(\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 3 \end{array}\right)$$

Thus $x_1 = -8$ *and* $x_2 = 3$

Let's reproduce these computation using the augmented matrix $(A|I)$ we write the **augmented matrix** $(A|x)$.

$$\left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ -2 & -7 & 0 & 1 \end{array}\right)$$

Then $2R_1 + R_2 \to R_2$

$$\left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 3 & 2 & 1 \end{array}\right)$$

Then $R_2/3$

$$\left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 1 & 2/3 & 1/3 \end{array}\right)$$

Then $-5R_2 + R_1 \to R_1$

$$\left(\begin{array}{cc|cc} 1 & 0 & -7/3 & -5/3 \\ 0 & 1 & 2/3 & 1/3 \end{array}\right)$$

Exercise 6. *Verify that*

$$A^{-1} = \begin{pmatrix} -7/3 & -5/3 \\ 2/3 & 1/3 \end{pmatrix}$$

and that

$$A^{-1} \begin{pmatrix} 7 \\ -5 \end{pmatrix} = \begin{pmatrix} -8 \\ 3 \end{pmatrix}$$

Exercise 7. *Find the solution to the linear system*

$$\begin{array}{rrrcr} & x_2 & +x_3 & = & -8 \\ x_1 & -2x_2 & -3x_2 & = & 0 \\ -x_1 & +x_2 & 2x_3 & = & 3 \end{array}$$

3 Determinants

We will consider the concept of the determinant in the case of 2×2 and 3×3 matrices. A square matrix C of any dimension is invertible if and only if its determinant $\det(C) \neq 0$.

For a 2×2 matrix

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\det(C) = ad - bc$ and the matrix inverse

$$C^{-1} = \frac{1}{\det(C)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

We can verify this using the Gauss-Jordan algorithm.

Exercise 8. *Use the formula above to obtain the inverse to*

$$A = \begin{pmatrix} 1 & 5 \\ -2 & -7 \end{pmatrix}$$

For a three by three matrix, the determinant is

$$c_{11} \det \begin{pmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{pmatrix} - c_{12} \det \begin{pmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{pmatrix} + c_{13} \det \begin{pmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}$$

For larger matrices, we have the same alternating form based on linear combinations of determinants from **minors**, matrices created by eliminating the first row and a column. Beyond its computational use, The determinant's value is summarized in the following theorem.

Theorem 9. *Let C be a $d \times d$ matrix. The following statements are equivalent:*

1. *C is singular (does not have an inverse).*
2. *The determinant of C is zero.*
3. *$Cx = 0$ has nontrivial solutions ($x = 0$).*
4. *The columns (rows) of C form a linearly dependent set.*