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Chapter 4.1: Functions, examples and problems

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Problems:
(1) Let A be the set of upper- and lowercase characters. Define $f : A^5 \to A^5$ by capitalizing the first letter (if necessary). What is the domain and range?
(2) Consider two functions $f, g : \mathbb{N} \to \mathbb{N}$, defined by $f(x) = x^3$ and $g(x) = |x|^3$. Is $f = g$? What if the domain and range of f, g was $\mathbb{Z}$ instead?

A function $f : X \to Y$ is

- one-to-one ("injective") if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.
- onto ("surjective") if for each $y \in Y$, there exists $x \in X$ so that $f(x) = y$. Note this says that the co-domain and range are the same.
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(2) Let $f(b_1 b_2 b_3 b_4) = b_1 b_2$. Is this 1-1 and onto?
Observe $f(1101) = 11 = f(1110)$, so not one-to-one. If $f(b_1 b_2 b_3 b_4) = a_1 a_2$, can have $b_1 = a_1$, $b_2 = a_2$, and b_3, b_4 arbitrary, so onto.

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(1) If not everyone got an A, let B be the set of those who did not. Then $f(B) = f(\phi)$.
(2) If everyone got an A, then for any pair of subsets B_1, B_2 of the same cardinality, $f(B_1) = f(B_2)$.

If f is onto, then there must be a subset of any size of students who got As. This can only happen if everyone got an A.

(1) Let X = set of countries, and Y = set of capital cities.
(a) Can $f : X \to Y$ be defined? What about $f : Y \to X$? (Note: Some places like the Netherlands have an official capital (Amsterdam) and an administrative capital (Hague)).
(b) If so, is $f : X \to Y$ 1-1 onto?

(2) Let $A = \{1, 2, 3\}$. Consider the function $f(X) : P(A) \to P(A)$ given by $f(X) = A - \{1\}$. Show this is not 1-1 or onto.

(3) Let $A = \{1, 2, 3\}$. Consider the function $f(X) : P(A) \to P(A)$ given by $f(X) = X \oplus \{1\}$. Show this IS 1-1 and onto.

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- Example: $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (2y, 3x)$. To find inverse, suppose $f(x, y) = (z, w)$ so that $z = 2y$ and $w = 3x$. Then $(x, y) = (w/3, z/2) = f^{-1}(w, z)$.

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(2) Let A, B be sets. Which are invertible, and what is the inverse?

$$f : A \times B \to A, \quad f(a, b) = a.$$

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- Inverses work backward: $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. (picture)

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Then $D \circ L : S \to \{T, F\}$ is given by $D(L(w))$.

(1) Given $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x) = 2x$ and $g(y) = y + 5$

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(2) A binary string of length 4 is encoded by first reversing the order of the digits, and then changing ones to zeros and zeros to ones.

(a) Using the notation $\overline{1} = 0$ and $\overline{0} = 1$, write expressions for the two separate encoding operations.

(b) Write the entire encoding as a composition of functions, and simplify this into a single function.

(c) Find the inverse of the composition.