Symmetries of the wave equation and special relativity

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Since $y = F(x + D/Ft)$, it may be considered to be a moving coordinate with speed $V = -D/F$.

Conditions for a linear transformation symmetry

Under what conditions is

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a symmetry? Changing variables can be accomplished using

$$u_t = Au_s + Du_y, \quad u_{tt} = A^2 u_{ss} + 2ADu_{ys} + D^2 u_{yy},$$

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The result is

$$(A^2 - c^2 B^2)u_{ss} + 2(AD - c^2 BF)u_{ys} = (c^2 F^2 - D^2)u_{yy},$$

In general, this equation does not have characteristic wave speeds $\pm c$.

Special relativity from symmetry

Can we choose A, B, C, D so that

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is still the standard wave equation? Need

$$AD = c^2 BF, \quad A^2 - c^2 B^2 = 1, \quad c^2 F^2 - D^2 = c^2,$$

which leads to (using $V = -D/F$)

$$A = \frac{1}{\sqrt{1 - V^2/c^2}} = F, \quad B = \frac{-V/c^2}{\sqrt{1 - V^2/c^2}} = Dc^2.$$

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What does this mean? Consider an event which occurs at (y_0, s_0), corresponding to (x_0, t_0), and another even which occurs at $(y_0, s_0 + \Delta s)$, corresponding to $(x_0 + \Delta x, t_0 + \Delta t)$. How much time has elapsed in one coordinates system versus the other?

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$$\Delta s = \frac{\Delta t - V(\Delta x)/c^2}{\sqrt{1 - V^2/c^2}} = \Delta t\sqrt{1 - V^2/c^2}.$$