One dimensional transport equations and the d'Alembert solution of the wave equation

Consider the simplest PDE: a first order, one dimensional equation

$$u_t + cu_x = 0 \tag{1}$$

on the entire real line $x \in (-\infty, \infty)$. This is sometimes called the transport equation, because it is the conservation law with the flux $-cu$, where c is the transport velocity. We can view (1) as the directional derivative of u in the direction $\mathbf{v} = (1, c)$ where $\mathbf{v}$ is a vector in (t, x)-space. Thus (1) means that the function $u(x, t)$ is **constant** on each line parallel to $\mathbf{v}$. These lines have the equation $x - ct =$ some constant, and therefore u must be a function of $x - ct$ alone. The most general solution of (1) is therefore

$$u(x, t) = f(x - ct). \tag{2}$$

If we were supplied with an initial condition to (1), we immediately find that $f(x) = u(x, 0)$. The solution (2) merely translates the initial data at speed c as time progresses.

Now consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

on the entire real line $x \in (-\infty, \infty)$. We can factor the linear operator to give

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

Setting $v = u_t + cu_x$, we get the two equations

$$\begin{aligned} v_t - cv_x &= 0, \\ u_t + cu_x &= v. \end{aligned}$$

(This is like solving the linear system $ABx = b$: we can introduce an intermediate quantity $w = Bx$ and solve $Aw = b$, then go back and solve $w = Bx$). The general solution to the first equation is just $v = h(x + ct)$ for some function h. Now we must solve

$$u_t + cu_x = h(x + ct). \tag{3}$$

This is an *inhomogeneous* equation, and we can attempt to solve it by looking for solutions of the form $u = u_{\rm hom} + u_p$ where u_p is a particular solution and $u_{\rm hom}$ solves the homogeneous equation

$$u_t + cu_x = 0. \tag{4}$$

The general solution to (4) is $u_{\rm hom} = g(x - ct)$. We guess a particular solution of the form $u_p = f(x + ct)$, and plugging it in gives $f'(s) = h(s)/2c$. In other words, the particular solution can also have the form $f(x + ct)$, so the whole solution $u = u_{\rm hom} + u_p$ has the general form

$$u = g(x - ct) + f(x + ct). \tag{5}$$

Now we would like to satisfy the initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x)$$

Inserting the general solution (5) gives

$$f(x) + g(x) \quad = u_0(x), \tag{6}$$
$$f'(x) - g'(x) \quad = \tfrac{1}{c} v_0(x). \tag{7}$$

Integrating the second of these gives

$$f(x) - g(x) = \frac{1}{c} \int_0^x v_0(x')dx' + K \tag{8}$$

where K is some constant of integration (the lower bound on the integral was specified arbitrarily). We can now add and subtract (6) and (8) in order to find f and g:

$$\begin{aligned} f(x) \quad &= \tfrac{1}{2} \left(u_0(x) + \tfrac{1}{c} \textstyle\int_0^x v_0(x')dx' + K \right), \\ g(x) \quad &= \tfrac{1}{2} \left(u_0(x) - \tfrac{1}{c} \textstyle\int_0^x v_0(x')dx' - K \right) \end{aligned}$$

Therefore the complete solution to the initial value problem is (notice that K drops out)

$$u = f(x+ct) + g(x-ct) = \frac{1}{2} \left(u_0(x + ct) + u_0(x - ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(x')dx'.$$

This is known as d'Alembert's solution to the wave equation.

The above method can be generalized to any second order PDE which can be factored and written as two transport equations. For example,

$$u_{xx} + u_{xy} - 2u_{yy} = 0 \tag{9}$$

can be factored as

$$\left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = 0.$$

which can be written as the system

$$\begin{aligned} v_x + 2v_y &= 0, \\ u_x - u_y &= v. \end{aligned}$$

Following the same logic as above, we see that the most general solution is

$$u(x, y) = f(y - 2x) + g(y + x)$$

for arbitrary functions f, g.