Dispersion relations, linearization and linearized dynamics in PDE models

1 Dispersion relations

Suppose that $u(x,t)$ is a function with domain $\{-\infty < x < \infty, t > 0\}$, and it satisfies a linear, constant coefficient partial differential equation. It happens that these type of equations have special solutions of the form

$$u(x,t) = \exp(ikx - i\omega t), \tag{1}$$

or equivalently,

$$u(x,t) = \exp(\sigma t + ikx). \tag{2}$$

We typically look for solutions of the first kind (1) when wave-like behavior which oscillates in time is expected, whereas (2) is used if we expect growth or decay in time. Plugging either (1) or (2) into the equation yields an algebraic relationship of the form $\omega = \omega(k)$ or $\sigma = \sigma(k)$, called the *dispersion relation*. It characterizes the dynamics of spatially oscillating modes of the form $\exp(ikx)$.

For dispersion relations of the form $\omega(k)$, notice that the solution can be written

$$u(x,t) = \exp\left(ik[x - \frac{\omega(k)}{k}t]\right), \tag{3}$$

which we notice are waves traveling at speed $\omega(k)/k$; this is known as the *phase velocity*. Of course, a more general solution might be a superposition of waves of the form (1), which means that there can be many different phase velocities present.

For dispersion relations of the form $\sigma(k)$, the sign of the real part of σ indicates whether the solution will grow or decay in time. If the real part of $\sigma(k)$ is negative for all k values, then any superposition of solutions of the form $\exp(\sigma t + ikx)$ will also appear to decay. On the other hand, if $\sigma(k) > 0$ for some values of k, then over time some components of a superposition will grow exponentially.

Here are a couple standard examples. For the wave equation $u_{tt} = c^2 u_{xx}$, we plug in a wave-like solution (1) to get $-\omega^2 \exp(ikx - i\omega t) = -c^2k^2 \exp(ikx - i\omega t)$, or $\omega(k) = \pm ck$. The phase velocity is $\omega/k = \pm c$, which happens to be a constant for all k. We might have expected this since the general solution of the wave equation has components which propagate at exactly these two velocities.

For the diffusion equation $u_t = Du_{xx}$, we use (2) to get $\sigma(k) = -Dk^2$. Since this is negative for all k, we might expect that solutions which are superpositions of (2) also decay. This is consistent with the fundamental solution representation for the diffusion equation.

2 Steady state solutions

Suppose we have a (possibly nonlinear) PDE of the form

$$u_t = R(u, u_x, u_{xx}, \ldots) \tag{4}$$

A *steady state* (or equilibrium) solution u_0 is one for which $(u_0)_t \equiv 0$, so that it solves

$$R(u_0, (u_0)_x, \ldots) = 0. \tag{5}$$

In addition, (4) and (5) might also satisfy boundary conditions. The steady state solution therefore solves a differential equation, although there are fewer independent variables than (4).

Often, the solution to (5) is just a constant $u = u_0$ in *space* as well as time. For example, for the diffusion equation with Dirichlet-type boundary conditions

$$u_t = u_{xx}, \quad u(0,t) = 2 = u(1,t), \tag{6}$$

it is easy to see that $u(x,t) = 2$ is a solution which does not depend on time or the space variable. In general, however, equilibria may depend on x; for example, for the diffusion equation with mixed boundary conditions

$$u_t = u_{xx}, \quad u(0,t) = 0, \quad u_x(1,t) = 1, \tag{7}$$

the equilibrium solution solves a two-point boundary value problem

$$(u_0)_{xx} = 0, \quad u_0(0) = 0, \quad (u_0)_x(1) = 1, \tag{8}$$

whose solution is easily obtained as $u_0 = x$.

3 Linearization and stability

It is frequently useful to approximate a nonlinear equation with a linear one, since we know a lot more about linear equations. If $u_0(x)$ is a steady state of an equation of the form (4), then we can look for solutions of the form

$$u(x,t) = u_0(x) + \epsilon w(x,t) \tag{9}$$

where ϵ presumed to be a small parameter. Plugging into (4) and keeping only the terms of order ϵ always gives us a linear, time dependent equation for w. This equation is called the *linearization* of (4) about u_0.

As an example, consider the "Fisher" (or KPP) equation with no-flux conditions at infinity

$$u_t = u_{xx} + u(1-u), \quad \lim_{x\to\pm\infty} u_x = 0. \tag{10}$$

An equilibrium solution $u = u_0(x)$ satisfies

$$(u_0)_{xx} + u_0(1-u_0) = 0, \quad \lim_{x\to\pm\infty} u_x = 0. \tag{11}$$

It is easy to pick off two constant (in space and time) solutions $u_0(x) = u_0$, which solve $u_0(1-u_0) = 0$ so that $u_0 = 0$ or $u_0 = 1$.

Now plugging (9) into (10) for $u_0 = 0$, one gets

$$\epsilon w_t = \epsilon w_{xx} + \epsilon w - \epsilon^2 w.$$

Keeping only terms of order ϵ, we get the linearized Fisher equation

$$w_t = w_{xx} + w. \tag{12}$$

This is a diffusion equation with a linear source term. It should be noted that $u_0 + \epsilon w$ only makes sense as an approximation to the original equation (11) if w is not too big. Nevertheless, a linearized equation often gives a lot of information about how the exact nonlinear equation behaves.

If we linearize about $u_0 = 1$, we get a different result. Setting $u = 1+\epsilon w$, we have

$$\epsilon w_t = \epsilon w_{xx} - \epsilon w - \epsilon^2 w,$$

so that the linearization is now

$$w_t = w_{xx} - w. \tag{13}$$

3.1 Stability

When we linearize about a constant-valued, steady state solution, we often end up with a constant coefficient linear equation. We can compute a dispersion relation of the form (2), which tells us whether solutions to the linearized equation grow or decay.

Consider the Fisher equation linearized about $u = 0$ in equation (12). Substituting (2) into it, one gets $\sigma = -k^2 + 1$. Since $\sigma > 0$ when $|k| < 1$, we might expect that an arbitrary initial condition for w has components that both grow and decay exponentially. We would therefore say that $u_0 = 0$ is *linearly unstable*. On the other hand, the linearization about $u = 1$ (13) has the dispersion relation $\sigma = -k^2 - 1$, which is always negative. Thus any initial condition of the form $u(x, 0) = 1 + \epsilon w(x, 0)$ in the original equation should evolve in time in a way that has w decay and $u \to 1$ as $t \to \infty$. We call this situation *linearly stable*.

4 Behavior of superpositions of waves

The dispersion relation for waves $\omega = \omega(k)$ says that each component of wavenumber k travels at the phase velocity $\omega(k)/k$. If the phase velocity is different for each k, a superposition of many different waves will appear to spread out or *disperse*. Surprisingly, if the superposition contains only wavenumbers near a central wavenumber k_0, the wave does not appear to move at the phase velocity $\sigma(k_0)/k_0$, but at a different speed. This is an effect of interference, because some waves move faster than others.

Consider an initial condition which is a superposition of many different oscillations

$$u(x, 0) = \int_{-\infty}^{\infty} A(k) e^{ikx} \, dk,$$

where we think of $A(k)$ as the amount of wavenumber k. If u solves an equation with a given dispersion relation $\omega = \omega(k)$ then the complete solution can be written

$$u(x, t) = \int_{-\infty}^{\infty} A(k) e^{ikx - i\omega(k)t} \, dk. \tag{14}$$

In general, the integral oscillates considerably as k is varied, and so one expects significant cancellation.

Suppose that $A(k)$ is concentrated about some wavenumber k_0; for example, $A(k) = \exp(-(k - k_0)^2/\epsilon)$ where ϵ is small. One might think that $A(k)$ could be replaced with a delta function $\delta(k - k_0)$ as a good approximation, and then one recovers $u \sim \exp(ik_0 x - i\omega(k_0)t)$. This turns out to be too simple, since the "sideband" wavenumbers near k_0 play a role as t becomes large. Instead, we Taylor expand $\omega(k) \approx \omega(k_0) + \omega'(k_0)(k - k_0)$, and plug into (14), giving

$$u(x, t) \approx e^{it[\omega'(k_0)k_0 - \omega(k_0)]} \int_{-\infty}^{\infty} A(k) e^{ik(x - \omega'(k)t)} \, dk.$$

The first factor just oscillates in time, whereas the integral is a traveling wave of the form $f(x - \omega'(k_0)t)$. This means that the entire superposition appears to travel at the *group velocity* $\omega'(k_0)$, which is in general different than the phase velocity $\omega(k_0)/k_0$. Only in the case where the dispersion relation is linear $\omega = ck$ do the phase and group velocities coincide.