Fourier transform techniques

1 The Fourier transform

Recall for a function $f(x) : [-L, L] \to \mathbb{C}$, we have the orthogonal expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(y)e^{-in\pi y/L}dy. \tag{1}$$

We think of c_n as representing the "amount" of a particular eigenfunction with wavenumber $k_n = n\pi/L$ present in the function $f(x)$. So what if L goes to ∞? Notice that the allowed wavenumbers become more and more dense. Therefore, when $L = \infty$, we expect $f(x)$ is a superposition of an uncountable number of waves corresponding to every wavenumber $k \in \mathbb{R}$, which can be accomplished by writing $f(x)$ as a integral over k instead of a sum over n.

Now let's take the limit $L \to \infty$ formally. Setting $k_n = n\pi/L$ and $\Delta k = \pi/L$ and using (1), one can write

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{-L}^{L} f(y)e^{-ik_n y}dy\right)e^{ik_n x}\Delta k.$$

Notice this is a Riemann sum for an integral on the interval $k \in (-\infty, \infty)$. Taking $L \to \infty$ is the same as taking $\Delta k \to 0$, which gives

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}dk, \tag{2}$$

where

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx. \tag{3}$$

The function $F(k)$ is the *Fourier transform* of $f(x)$. The *inverse transform* of $F(k)$ is given by the formula (2). (Note that there are other conventions used to define the Fourier transform). Instead of capital letters, we often use the notation $\hat{f}(k)$ for the Fourier transform, and $\check{F}(x)$ for the inverse transform.

1.1 Practical use of the Fourier transform

The Fourier transform is beneficial in differential equations because it can reformulate them as problems which are easier to solve. In addition, many transformations can be made simply by applying predefined formulas to the problems of interest. A small table of transforms and some properties is given below. Most of these result from using elementary calculus techniques for the integrals (3) and (2), although a couple require techniques from complex analysis.

A Brief table of Fourier transforms

Description	Function	Transform
Delta function in x	$\delta(x)$	1
Delta function in k	1	$2\pi\delta(k)$
Exponential in x	$e^{-a\|x\|}$	$\frac{2a}{a^2+k^2} \quad (a>0)$
Exponential in k	$\frac{2a}{a^2+x^2}$	$2\pi e^{-a\|k\|} \quad (a>0)$
Gaussian	$e^{-x^2/2}$	$\sqrt{2\pi}e^{-k^2/2}$
Derivative in x	$f'(x)$	$ikF(k)$
Derivative in k	$xf(x)$	$iF'(k)$
Integral in x	$\int_{-\infty}^{x} f(x')dx'$	$F(k)/(ik)$
Translation in x	$f(x-a)$	$e^{-iak}F(k)$
Translation in k	$e^{iax}f(x)$	$F(k-a)$
Dilation in x	$f(ax)$	$F(k/a)/a$
Convolution	f(x)*g(x)	$F(k)G(k)$

Typically these formulas are used in combination. Preparatory steps are often required (just like using a table of integrals) to obtain exactly one of these forms. Here are a few examples.

Example 1. The transform of $f''(x)$ is (using the derivative table formula)

$$\left[f''(x)\right]^{\wedge} = ik\left[f'(x)\right]^{\wedge} = (ik)^2\hat{f}(k) = -k^2\hat{f}(k).$$

Notice what this implies for differential equations: differential operators can be turned into "multiplication" operators.

Example 2. The transform of the Gaussian $\exp(-Ax^2)$ is, using both the dilation and Gaussian formulas,

$$\begin{aligned}\left[\exp(-Ax^2)\right]^{\wedge} &= \left[\exp(-[\sqrt{2A}x]^2/2)\right]^{\wedge} = \frac{1}{\sqrt{2A}}\left[\exp(-x^2/2)\right]^{\wedge}(k/\sqrt{2A}) \\ &= \sqrt{\frac{\pi}{A}}\exp(-[k/\sqrt{2A}]^2/2) = \sqrt{\frac{\pi}{A}}\exp(-k^2/(4A)).\end{aligned}$$

Example 3. The inverse transform of $e^{2ik}/(k^2+1)$ is, using the translation in x property and then the exponential formula,

$$\left(\frac{e^{2ik}}{k^2+1}\right)^{\vee} = \left(\frac{1}{k^2+1}\right)^{\vee}(x+2) = \frac{1}{2}e^{-|x+2|}.$$

Example 4. The inverse transform of $ke^{-k^2/2}$ uses the Gaussian and derivative in x formulas:

$$\begin{aligned}&\left[ke^{-k^2/2}\right]^{\vee} = -i\left[ike^{-k^2/2}\right]^{\vee} = -i\frac{d}{dx}\left[e^{-k^2/2}\right]^{\vee} = \\ &= \frac{-i}{\sqrt{2\pi}}\frac{d}{dx}\left[\sqrt{2\pi}e^{-k^2/2}\right]^{\vee} = \frac{-i}{\sqrt{2\pi}}\frac{d}{dx}\left(e^{-x^2/2}\right) = \frac{ix}{\sqrt{2\pi}}e^{-x^2/2}.\end{aligned}$$

1.2 Convolutions

Unfortunately, the inverse transform of a product of functions is not the product of inverse transforms. Rather, it is a binary operation called *convolution*, defined as

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy. \tag{4}$$

Using the definition, the Fourier transform of this is

$$(f * g)^\wedge = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)g(y)e^{-ikx} dy\, dx$$

Using the change of variables $z = x - y$, this becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z)g(y)e^{-ik(y+z)} dydz = \left(\int_{-\infty}^{\infty} f(z)e^{-ikz} dz \right) \left(\int_{-\infty}^{\infty} g(y)e^{-iky} dy \right) = \hat{f}(k)\hat{g}(k),$$

which is just the last formula in the table.

1.3 Divergent Fourier integrals as distributions

The formulas (3) and (2) assume that $f(x)$ and $F(k)$ decay at infinity so that the integrals converge. If this is not the case, then the integrals must be interpreted in a generalized sense. In addition, some of the table formulas must be adjusted to take this into account.

Notice that the transform of $\delta(x)$ equals $\hat{f}(k) \equiv 1$, so at least formally, the inverse transform of $\hat{f}(k)$ should be a delta function:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk. \tag{5}$$

This is very troublesome: the integral does not even converge, so what could such a statement mean?

Of course the issue is that the integral represents a distribution, not a regular function. We can define the inverse transform of $F(k)$ more generally as a distribution which is the limit of the regular functions

$$f_L(x) = \frac{1}{2\pi} \int_{-L}^{L} \exp(ikx) F(k) dk$$

as $L \to \infty$ (recall the fact that distributions can always be approximated by regular functions). This means that the inverse transform $f(x)$ is a distribution which acts on smooth functions like

$$f[\phi] = \lim_{L\to\infty} \int_{-\infty}^{\infty} f_L(x)\phi(x) dx. \tag{6}$$

Let's look at the integral in (5) and see what distribution it represents. For this case,

$$f_L(x) = \frac{1}{2\pi} \int_{-L}^{L} \exp(ikx) dk = \frac{1}{\pi x} \sin(Lx).$$

Then if one asks how the limit of f_L acts as a distribution, one computes

$$f[\phi] = \lim_{L\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(Lx)}{x} \phi(x) dx = \lim_{L\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(y)}{y} \phi(y/L) dy.$$

The limit $L \to \infty$ can be taken inside the integral (this can be justified with a little effort), and this results in

$$f[\phi] = \frac{\phi(0)}{\pi} \int_{-\infty}^{\infty} \frac{\sin(y)}{y} dy = \phi(0).$$

So the inverse transform really is the delta function!

2 Solutions of differential equations using transforms

The derivative property of Fourier transforms is especially appealing, since it turns a differential operator into a multiplication operator. In many cases this allows us to eliminate the derivatives of one of the independent variables. The resulting problem is usually simpler to solve. Of course, to recover the solution in the original variables, an inverse transform is needed. This is typically the most labor intensive step.

2.1 Ordinary differential equations on the real line

Here we give a few preliminary examples of the use of Fourier transforms for differential equations involving a function of only one variable.
Example 1. Let us solve

$$-u'' + u = f(x), \qquad \lim_{|x|\to\infty} u(x) = 0. \tag{7}$$

The transform of both sides of (7) can be accomplished using the derivative rule, giving

$$k^2 \hat{u}(k) + \hat{u}(k) = \hat{f}(k). \tag{8}$$

This is just an algebraic equation whose solution is

$$\hat{u}(k) = \frac{\hat{f}(k)}{1+k^2}. \tag{9}$$

We can recover $u(x)$ by an inverse transform. Expression (9) is a product of $\hat{f}(k)$ and $1/(1+k^2)$, so we must use the convolution formula:

$$u(x) = f(x) * \left(\frac{1}{1+k^2}\right)^{\vee} = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)dy.$$

This is exactly what we get from a Green's function representation, where $G(x,y) = e^{-|x-y|}/2$. There is one mystery remaining: the far field condition in (7) does not seem to be used anywhere. In fact, condition (7) is already built into the Fourier transform; if the functions being transformed did not decay at infinity, the Fourier integral would only be defined as a distribution as in (6).
Example 2. The *Airy* equation is

$$u'' - xu = 0,$$

which will be subject to the same far field condition as in (7). The transform uses the derivative formulas for both x and k, giving

$$-k^2\hat{u}(k) - i\hat{u}'(k) = 0.$$

This is still a differential equation in the k variable, but we can solve it by separation of variables (the ODE version, that is). This results in $d\hat{u}/\hat{u} = ik^2 dk$ which integrates to give

$$\hat{u}(k) = Ce^{ik^3/3},$$

where C is an arbitrary constant of integration. The inverse transform is

$$u(x) = \frac{C}{2\pi}\int_{-\infty}^{\infty} \exp(i[kx + k^3/3])dk. \tag{10}$$

This integral cannot be reduced any further. With the choice $C = 1$, the result is the so-called *Airy function* denoted $\mathrm{Ai}(x)$.

2.2 Solution of partial differential equations

Now we consider situations where there is more than one independent variable. In this case, the transform will apply to only one variable. This will reduce the number of variables which have derivatives, and often make it possible to solve using ODE techniques.
Example 1. Consider the Laplace equation on the upper half plane

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0, \quad u(x,0) = g(x), \quad \lim_{y\to\infty} u(x,y) = 0. \tag{11}$$

It only makes sense to transform in the x variable, and we denote this as

$$U(k,y) = \int_{-\infty}^{\infty} e^{-ikx} u(x,y)dx. \tag{12}$$

We note that the y-derivative commutes with the Fourier integral in x, so that the transform of u_{yy} is simply U_{yy}. Then the equation and boundary conditions in (11) become

$$-k^2 U + U_{yy} = 0, \quad U(k,0) = \hat{g}(k), \quad \lim_{y\to\infty} U(k,y) = 0.$$

This is a set of ordinary differential equations, one for each value of k. The general solution is $U = c_1 e^{+|k|y} + c_2 e^{-|k|y}$, where $c_{1,2}$ can depend on k. The first term must be zero so that U vanishes at infinity. Using $U(k,0) = \hat{g}(k)$, it follows that

$$U(k,y) = \hat{g}(k)e^{-|k|y}.$$

The inverse transform involves a convolution and the exponential in k formula from the table. The result is

$$\begin{aligned} u(x,y) =& g(x) * \left(e^{-|k|y}\right)^{\vee} = g(x) * \left(\frac{y}{\pi(x^2+y^2)}\right) \\ =& \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y g(x_0)}{(x-x_0)^2 + y^2} dx_0. \end{aligned}$$

This is precisely the same formula as obtained by Green's function methods.
Example 2. Now let's solve the transport equation

$$u_t + c u_x = 0, \quad -\infty < x < \infty, \quad t > 0, \qquad u(x,0) = f(x)$$

by a similar process. Let $U(k,t)$ be the transform of u only in the x variable as in (12). Since the t derivative commutes with the x-integral, the problem transforms into

$$U_t + ikcU = 0, \quad U(k,0) = \hat{f}(k).$$

This is a simple first order differential equation whose solution is

$$U(k,t) = e^{-ickt}\hat{f}(k).$$

Now we use the translation formula from the table with $a = ct$, which means that the inverse transform is

$$u(x,t) = f(x - ct).$$

This is a *traveling wave solution*, describing a pulse with shape $f(x)$ moving uniformly at speed c.

Example 3. Consider the wave equation on the real line

$$u_{tt} = u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Letting $U(k,t)$ be the transform in the x-variable, then the problem becomes

$$U_{tt} + k^2 U = 0, \quad U(k,0) = \hat{f}(k), \quad U_t(k,0) = \hat{g}(k).$$

This is just the initial value problem for a harmonic oscillator. Its solution is

$$U(k,t) = \hat{f}(k)\cos(kt) + \frac{\hat{g}(k)}{k}\sin(kt). \tag{13}$$

Note that sines and cosines can be written in terms of complex exponentials, so that

$$U(k,t) = \frac{1}{2}\hat{f}(k)(e^{ikt} + e^{-ikt}) + \frac{1}{2ik}\hat{g}(k)(e^{ikt} - e^{-ikt}). \tag{14}$$

The inverse transform is now straightforward, using the exponential and integral formulas,

$$u(x,t) = \frac{1}{2}[f(x-t) + f(x+t)] + \frac{1}{2}\int_{-\infty}^{x} g(x'+t) - g(x'-t)dx'.$$

The integral can be simplified by using the change of variables $\xi = x' + t$ for the first term and $\xi = x' - t$ for the second,

$$\int_{-\infty}^{x} g(x'+t) - g(x'-t)dx' = \int_{-\infty}^{x+t} g(\xi)d\xi - \int_{-\infty}^{x-t} g(\xi)d\xi = \int_{x-t}^{x+t} g(\xi)d\xi.$$

Finally, putting this all together results in *d'Alembert's* formula for the wave equation

$$u(x,t) = \frac{1}{2}[f(x-t) + f(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)d\xi. \tag{15}$$

3 Fundamental solutions for time dependent equations

Partial differential equations which involve time also have Green's functions, although they are more often called *fundamental solutions* or *source functions*. Suppose that $u(\mathbf{x},t) : D \times \mathbb{R} \to \mathbb{R}$, where $D \subset \mathbb{R}^n$ is some spatial domain, which solves

$$u_t(\mathbf{x},t) = \mathcal{L}u(\mathbf{x},t), \quad u(\mathbf{x},0) = f(\mathbf{x}), \tag{16}$$

where $\mathcal{L}$ is some differential operator which does not depend on t. This equation is supplemented by homogeneous boundary conditions (Dirichlet, Neumann and possibly others) on $\mathbf{x} \in \partial D$.

We define the fundamental solution for (16) to be the solution $S(\mathbf{x},\mathbf{x}_0,t)$ to the problem

$$S_t = \mathcal{L}_x S, \quad S(\mathbf{x},\mathbf{x}_0,0) = \delta(\mathbf{x}-\mathbf{x}_0),$$

subject to the same boundary conditions as u.

This is different than our previous definition of Green's function in one respect: the δ-function appears as an initial condition rather than an inhomogeneous term in the equation. Of course, the initial condition on S is only meaningful in the sense of distributions, which is to say that S limits to a δ-function as $t \to 0$:

$$\lim_{t \to 0} \int_D S(\mathbf{x}, \mathbf{x}_0, t)\phi(\mathbf{x})dx = \int_D \delta(\mathbf{x} - \mathbf{x}_0)\phi(\mathbf{x})dx = \phi(\mathbf{x}_0), \tag{17}$$

for all continuous functions $\phi : D \to \mathbb{R}$.

We now claim that the initial value problem (16) is solved by the formula

$$u(x,t) = \int_{-\infty}^{\infty} S(\mathbf{x}, \mathbf{x}_0, t) f(\mathbf{x}_0)dx_0, \tag{18}$$

so long as f is continuous. The intuitive meaning is clear: the integral is the just a sum over all point influences contained in the initial condition $f(\mathbf{x})$. Let's check to see that it works. For the initial condition, we take $t \to 0$ in (18) and use (17),

$$u(x,0) = \lim_{t \to 0} \int_D S(\mathbf{x}, \mathbf{x}_0, t) f(\mathbf{x}_0)dx_0 = \int_D \delta(\mathbf{x} - \mathbf{x}_0) f(x_0)dx_0 = f(\mathbf{x}).$$

Plugging u into the equation in (16), we can move the time derivative inside the integral:

$$u_t = \int_D S_t(\mathbf{x}, \mathbf{x}_0, t) f(\mathbf{x}_0)dx_0 = \int_D \mathcal{L}_x S(\mathbf{x}, \mathbf{x}_0, t) f(\mathbf{x}_0)dx_0.$$

We now suppose that the operator (which acts on the $\mathbf{x}$ variables) can be moved outside the $\mathbf{x}_0$ integral (this can be justified in most circumstances), giving

$$u_t = \mathcal{L}_x \int_D S(\mathbf{x}, \mathbf{x}_0, t) f(\mathbf{x}_0)dx_0 = \mathcal{L}_x u,$$

which means the equation is satisfied.

3.1 Finding fundamental solutions with the Fourier transform

When the spatial domain is $D = \mathbb{R}$, the Fourier transform can often be used to find fundamental solutions.

Example 1. For the diffusion equation on the real line

$$u_t = Du_{xx}, \quad -\infty < x < \infty, \quad u(x,0) = f(x), \lim_{|x| \to \infty} u(x,t) = 0.$$

the fundamental solution solves

$$S_t = DS_{xx}, \quad -\infty < x < \infty, \quad S(x, x_0, 0) = \delta(x - x_0), \lim_{|x| \to \infty} S(x, x_0, t) = 0.$$

Taking the Fourier transform in x by letting

$$\hat{S}(k, x_0, t) = \int_{-\infty}^{\infty} S(x, x_0, t) e^{-ikx} dx,$$

we find that $\hat{S}$ solves a simple first order ODE, $\hat{S}_t = -Dk^2\hat{S}$. The initial condition for this ODE is given by the transform of $\delta(x - x_0)$, which by the translation property is e^{-ix_0k} multiplied by the transform of $\delta(x)$ which equals one. It follows that $\hat{S} = e^{-ix_0k - Dk^2t}$. The inverse transform uses the translation and dilation properties together with the transform of the Gaussian, giving

$$S(x, x_0, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/(4Dt)}. \tag{19}$$

If we wished to solve the diffusion equation $u_t = Du_{xx}$ on the real line subject to the initial condition $u(x, 0) = f(x)$, we can use the template (18), giving

$$u(x, t) = \int_{-\infty}^{\infty} \frac{f(x_0)}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/(4Dt)} dx_0. \tag{20}$$

Note this is the same as (20).
Example 2. The famous Korteweg - de Vries (KdV) equation can be approximated by the linear initial value problem

$$u_t = -u_{xxx}, \quad u(x, 0) = f(x), \quad \lim_{|x|\to\infty} u(x, t) = 0. \tag{21}$$

The corresponding fundamental solution solves

$$S_t = -S_{xxx}, \quad -\infty < x < \infty, \quad S(x, x_0, 0) = \delta(x - x_0), \lim_{|x|\to\infty} S(x, x_0, t) = 0.$$

Transforming as in the previous example, we get the initial value problem

$$\hat{S}_t = ik^3\hat{S}, \quad \hat{S}(k, 0) = e^{-ix_0k},$$

whose solution is $\hat{S}(k, t) = e^{-ix_0k}e^{ik^3t}$. Recall that the inverse transform of e^{ik^3} is the Airy function (10), so it follows that

$$S(x, x_0, t) = \left[e^{-ix_0k}e^{ik^3t}\right]^{\vee} = \left[\exp^{i(k/a)^3}\right]^{\vee}(x - x_0) = a\mathrm{Ai}\Big(a(x - x_0)\Big), \quad a \equiv (3t)^{-1/3}.$$

The solution of (21) can therefore be written

$$u(x, t) = \frac{1}{(3t)^{1/3}} \int_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{x - y}{(3t)^{1/3}}\right) f(y)dy.$$

3.2 The method of images for fundamental solutions

From our previous discussion on Green's functions, we know that symmetry may be used in cases where boundaries are present. For problems on the half-line $x > 0$ with boundary conditions $u(0, t) = 0$ or $u_x(0, t)$, the odd (or even) symmetric reflection of the fundamental solutions for the whole line will satisfy the correct boundary conditions.

Take, for example, the diffusion equation

$$u_t = Du_{xx}, \quad u(x, 0) = f(x),\ u(0, t) = 0,\ \lim_{x\to\infty} u(x, t) = 0. \tag{22}$$

We clearly can't use the Fourier transform, since the domain in x is not the entire real line, and the fundamental solution (19) doesn't have the correct boundary condition. On the other hand, the odd reflection (19)

$$S(x, x_0, t) = \frac{1}{\sqrt{4\pi Dt}} \left[e^{-(x-x_0)^2/(4Dt)} - e^{-(x+x_0)^2/(4Dt)}\right]$$

does get the boundary condition right. In addition, by superposition $S_t = S_{xx}$ and $S(x, x_0, 0) = \delta(x - x_0) - \delta(x + x_0)$. As usual in the method of images, the delta function $\delta(x + x_0)$ is not in the domain and can be ignored. Therefore the solution of (22) is just

$$u(x,t) = \int_0^\infty \frac{f(x_0)}{\sqrt{4\pi Dt}} \left[e^{-(x-x_0)^2/(4Dt)} - e^{-(x+x_0)^2/(4Dt)} \right] dx_0. \tag{23}$$