Green's function challenge: finding the toxic leak

Three underwater petroleum wells are located near the shore. One or perhaps more of them has been leaking a toxic substance, and examining the wells directly is an expensive undertaking.

1 The model

The concentration of the toxic substance $u(x, y, t)$ can be modeled assuming: the flux is diffusive so that $\mathbf{J} = -D\nabla u$, a chemical reaction removes the substance at a rate proportional to the amount (i.e. exponential decay), and th wells produce an unknown source $Q(x, y)$. This leads to the inhomogeneous diffusion equation

$$u_t = \Delta u - u + Q(x, y).$$

The shoreline is regarded as an infinite boundary and gives the no-flux boundary condition

$$u_y(x, 0, t) = 0.$$

The source term is modeled as a set of point sources

$$Q = \sum_{j=1}^{3} \sigma_j \delta(\mathbf{x} - \mathbf{s}_j)$$

where $\mathbf{s}_1 = (-1, 1)$, $\mathbf{s}_2 = (0, 2)$, and $\mathbf{s}_3 = (1, 1)$. In the long run, time can be ignored so that $u = u(x, y)$. Write down the inhomogeneous Poisson equation and boundary conditions that $u(x, y)$ will satisfy.

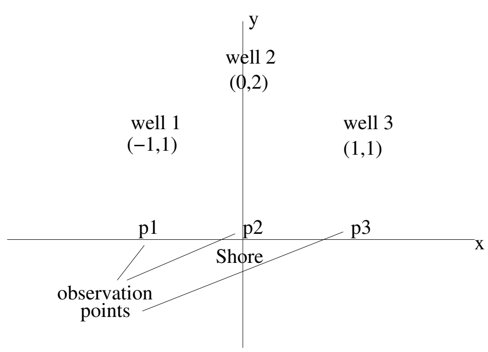

Figure 1: Location of the wells and observation points.

2 Green's function solution

Use the method of images to write down the Green's function for the problem in part I. Recall the two dimensional free-space Green's function for $\Delta u - u$ (the Helmholtz operator) is

$$G_\infty(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{2\pi} K_0(|\mathbf{x} - \mathbf{x}_0|)$$

where K_0 is a modified Bessel function. Then use this result to express your answer for $u(x, y)$ in closed form.

3 Solving the inverse problem

Instead of directly observing the wells, a series of measurements at points $\mathbf{p}_1 = (-1, 0)$, $\mathbf{p}_2 = (0, 0)$, and $\mathbf{p}_3 = (1, 0)$ are made along the shore to determine the concentration. It is found that $u(\mathbf{p}_1) = 0.0680$, $u(\mathbf{p}_2) = 0.1106$ and $u(\mathbf{p}_3) = 0.1214$. Use this information to determine the values of σ_j. Which well(s) are leaking?