

FAQ about the proof of the *abc*-conjecture

Kirti Joshi

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1 Introduction

This is a document providing answers to Frequently Asked Questions (FAQ) regarding the proof of the *abc*-conjecture given in [Joshi, 2021, 2022, 2023b,a, 2024c], [Mochizuki, 2021b].

Question 1. What do I need to know to understand the outlines of the claimed proof of the *abc*-conjecture?

Ans.

- (1.1) Read some classical Teichmüller Theory of genus one Riemann surfaces from any standard text on the subject (e.g. [Imayoshi and Taniguchi, 1992]). [A good understanding of the genus one case is essential.]
- (1.2) Next read the proof of the Geometric Szpiro Conjecture given in [Zhang, 2001] (its approach is simpler than the prior work of [Amorós, Bogomolov, Katzarkov, and Pantev, 2000]).
- (1.3) Understanding Teichmüller Theory aspects of the above cited papers is very important. My commentary on this appears in [Joshi, 2023a, Appendix] and it shows how the principal constructs of the arithmetic case appear in the geometric case. Mochizuki's commentary on the geometric case and his own arithmetic approach is here [Mochizuki, 2016]—this is a bit sophisticated, and it does not cover aspects covered in my commentary.
- (1.4) Finally read [Kim, 2002] (or [Beauville, 2002]) which presents a different proof of the Geometric Szpiro Conjecture.

Understanding the arithmetic case minimally requires a good understanding of classical Teichmüller Theory, [Zhang, 2001] and [Kim, 2002].

Question 2. Why should I be familiar with the geometric case?

Ans. Over the past hundred years, the development of proofs in Diophantine geometry has often proceeded as follows: understand and prove the geometric case, and then imitate or transfer ideas of that proof to the arithmetic case. A classic example of this is the proof of Mordell's Conjecture. My commentary on the geometric case (cited above) demonstrates how closely the proof in the geometric case parallels what Mochizuki and I do in the arithmetic case.

Question 3. I know a lot of algebraic geometry and moduli of Riemann surfaces. Is that not enough?

Ans. No. Teichmüller Theory deals with quasi-conformal mappings (e.g. $\mathbb{C} \xrightarrow{x+iy \mapsto x+5iy} \mathbb{C}$) between Riemann surfaces and these are not holomorphic and therefore not algebraic. Teichmüller Theory of Riemann surfaces extends Riemann's moduli theory of Riemann surfaces by allowing non-holomorphic mappings between Riemann surfaces.

Question 4. What is the relevance of Teichmüller Theory in the proof?

Ans. Mochizuki's first key idea is to extend the familiar analogy between Riemann Surfaces and Number Fields, and envisages a Teichmüller Theory for any fixed number field (my paper [Joshi, 2023a] independently demonstrates this idea and elevates it further to construct an Arithmetic Teichmüller Space for each fixed number field). Mochizuki's second key idea (very roughly speaking) is to average over the Teichmüller Theory of a Number Field (in [Joshi, 2024c], [Joshi, 2024b] the averaging is over the Arithmetic Teichmüller Space).

Question 5. Is there any interesting example of averaging techniques used in Teichmüller Theory of Riemann surfaces?

Ans. Yes. Averaging i.e. computing integrals over Teichmüller spaces is a standard technique. The survey [Wright, 2019] has some very interesting recent examples including the higher genus case.

Question 6. I know the analogy between Number Fields and Riemann surfaces very well. So how can I understand the extended Teichmüller analogy you are proposing?

Ans. Number Field is a global object. So before understanding the global arithmetic situation (documented in [Joshi, 2023a]), I recommend understanding the p -adic fields case first (i.e. the case of finite extensions of $\mathbb{Q}_p$).

Question 7. Are you suggesting that there is a Teichmüller Theory for p -adic fields?

Ans. Yes.

Question 8. I have studied p -adic fields and p -adic analysis from [Koblitz, 1984] very well, but I have never heard of this. So what is it?

Ans. In some sense, the p -adic field case began a long time ago ([Schmidt, 1933], [Kaplansky, 1942]). Let me give a quick overview using the point of view of my papers. Let $\mathbb{C}_p$ be the completion of some algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. By [Kedlaya and Temkin, 2018], there exists an algebraically closed, complete valued field K containing an isometric embedding of $\mathbb{Q}_p$, such that $K \simeq \mathbb{C}_p$ as abstract algebraically closed fields (so they have the same cardinality) and $\mathbb{C}_p$ have the same value group i.e. $|\mathbb{C}_p^*| = |K^*|$, and both have the same residue field $\bar{\mathbb{F}}_p$, but K and $\mathbb{C}_p$ are not topologically isomorphic and hence not isomorphic as (complete) valued fields. This means $K, \mathbb{C}_p$ each provides (a) an algebraic closure $\bar{\mathbb{Q}}_p$, (b) a completion of this algebraic closure $\bar{\mathbb{Q}}_p$, and (c) a valid version of p -adic analysis (as in [Koblitz, 1984]). But, there is no way to compare the algebraic closures or their completions as fields embedded in K and $\mathbb{C}_p$ or the p -adic analysis K and $\mathbb{C}_p$ provide.

Thus, p -adic arithmetic in K looks quite different relative to that in $\mathbb{C}_p$. So one may think of K as providing a deformation of arithmetic of $\mathbb{Q}_p, \bar{\mathbb{Q}}_p$ relative to that in $\mathbb{C}_p$.

My view of the p -adic field case was heavily influenced by [Schmidt, 1933], [Kaplansky, 1942]. To relate to the view of [Schmidt, 1933], the fields K and $\mathbb{C}_p$ are $\mathbb{Q}$ -isomorphic, but have inequivalent (rank one) valuations each of which induce a valuation equivalent to the standard p -adic valuation on the prime subfield $\mathbb{Q}$.

The fields $K, \mathbb{C}_p$ are both algebraically closed perfectoid fields containing $\mathbb{Q}_p$ isometrically, so it is natural to bring in [Scholze, 2012], [Fargues and Fontaine, 2018]. The view of [Joshi, 2023a] is that Fargues-Fontaine curves constructed in [Fargues and Fontaine, 2018] are the Teichmüller space for finite extensions of $\mathbb{Q}_p$.

(In the p -adic context, I should also mention [Matignon and Reversat, 1984], [Poonen, 1993]).

Question 9. I understand $\mathbb{C}_p$, but I don't understand K . Can't one work at the level of finite extensions of $\mathbb{Q}_p$?

Ans. No. If M_1, M_2 are finite extensions of $\mathbb{Q}_p$, then any field isomorphism $M_1 \simeq M_2$ is an isomorphism of complete discretely valued fields. So the situation is very rigid at the level of finite extensions of $\mathbb{Q}_p$.

Question 10. But what does this have to do with elliptic curves?

Ans. Let $E/\mathbb{Q}_p$ be an elliptic curve and $K, \mathbb{C}_p$ be as discussed above. In [Joshi, 2021], I show that the Berkovich analytic spaces E_K^{an} and $E_{\mathbb{C}_p}^{an}$ are not $\mathbb{Q}_p$ -isomorphic ($\mathbb{Q}_p$ is the smallest complete valued field isometrically embedded in both K and $\mathbb{C}_p$). Since Berkovich analytic spaces are built using (p -adic) analytic functions, one can think of $E_K^{an}, E_{\mathbb{C}_p}^{an}$ as analytic spaces arising from the elliptic curve $E/\mathbb{Q}_p$ but with completely distinct holomorphic structures. This deformation or change of holomorphic structures is achieved by changing p -adic arithmetic in $\mathbb{C}_p$ with p -adic arithmetic in K . That is why I have asserted that one is dealing with Arithmetic Teichmüller Theory. [Mochizuki coined the phrase “arithmetic holomorphic structure.”]

Question 11. But there is no mention of any field like K (or even $\mathbb{C}_p$) in [Mochizuki, 2021b]. So why is your claim related to Mochizuki's claim?

Ans. Mochizuki's theory is founded on his *Key Principle of Inter-Universality* [Mochizuki, 2021b, § I3, Page 25-26] (see Question 31). This principle requires one to work with arbitrary geometric base-points for tempered fundamental groups. This is how fields like $K, \mathbb{C}_p$ enter Mochizuki's work.

Question 12. I see that such large fields exist. But can you be more specific about how this Teichmüller theory of arithmetic affects an elliptic curve?

Ans. Before clarifying this let me explain how classical Teichmüller Theory impacts elliptic curves from an algebro-geometric perspective. In classical Teichmüller Theory, periods of a fixed Riemann surface get scaled arbitrarily (the isomorphism class of the Riemann surface remains fixed). In genus one, an elliptic curve E_τ with period lattice $[1, \tau]$ is isomorphic, via the Möbius transformation $\tau \mapsto -\frac{1}{\tau}$, to the elliptic curve $E_{-\frac{1}{\tau}}$ with periods lattice $[1, -\frac{1}{\tau}]$. If the original period τ satisfies $|\tau| > 1$, then $\tau' = -\frac{1}{\tau}$ satisfies $|\tau'| < 1$. This sort of relative scaling is a typical hallmark of Teichmüller Theory of Riemann surfaces. In genus one, classical

Teichmüller space can be identified with the complex upper half-plane, and the elliptic curves E_τ and $E_{-\frac{1}{\tau}}$ represent distinct points of the Teichmüller space, but they represent the same point in the moduli space (identified with the complex j -line).

Mochizuki asserted, that there is an arithmetic version of this relative Teichmüller type scaling of the Tate parameters i.e. the p -adic periods of a semi-stable elliptic curve. But, using Mochizuki's group-theoretic methods and tools, it is genuinely difficult to prove that this relative scaling exists in the p -adic case and is non-trivial. From my point of view this relative scaling exists simply because the valuations of all the large fields K , which enter the theory via geometric base-points (for tempered fundamental groups), cannot be simultaneously normalized. So the Tate parameters i.e. p -adic periods do occur at distinct valuation scales relative to each other.

Question 13. Can you be a bit more specific about the p -adic period aspect, because I do not understand this point.

Ans. Let $E/\mathbb{Q}_p$ be a semi-stable elliptic curve. Then the untilts $(K, K^b \simeq \mathbb{C}_p^b)$ are parameterized by the closed classical points of a suitable Fargues-Fontaine curve and each untilt $(K, K^b \simeq \mathbb{C}_p^b)$ provides a Tate parameter $q(K, K^b \simeq \mathbb{C}_p^b)$ and the rule

$$(K, K^b \simeq \mathbb{C}_p^b) \longmapsto q(K, K^b \simeq \mathbb{C}_p^b)$$

is a highly non-trivial and non-constant function on the Fargues-Fontaine curve. This is the function which one would like to average or integrate over the Fargues-Fontaine curve (or some chosen subset of it). The idea is, roughly speaking, to use this average to bound the Tate parameter $q(\mathbb{C}_p, \mathbb{C}_p^b = \mathbb{C}_p^b)$ at the canonical point $(\mathbb{C}_p, \mathbb{C}_p^b = \mathbb{C}_p^b)$ of the Fargues-Fontaine curve. This is the picture at one prime, but one really wants to do this over all primes and use the product formula as a global glue. Two remarks will be useful: first, integrals of similar functions occur in classical Teichmüller Theory ([Wright, 2019]). Second, in Grothendieck's function-sheaf dictionary, this rule should be seen as giving us a sheaf on the Fargues-Fontaine curve.

Question 14. You assert in Question 13 that product formula for number fields provides global gluing in the arithmetic case. But what plays the role of the product formula in geometric proof of [Zhang, 2001], [Amorós et al., 2000]?

Ans. The context for the geometric proofs in [Zhang, 2001] (and [Amorós et al., 2000]) is a non-isotrivial elliptic fibration $f : E \rightarrow B$ with stable fibers and with B a connected algebraic curve of genus $g \geq 0$ over complex numbers. Let $S \subset B$ be the set of points of B over which the fibers of f are not smooth. Let $|S| = n$ be the cardinality of S . Then let $\pi_1 = \pi_1(B - S)$ be the topological fundamental group of $B - S$, and let $a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_n$ be the generators of π_1 with the fundamental relation (in π_1)

$$\prod_{i=1}^g [a_i, b_i] \prod_{i=1}^n c_i = 1.$$

Then in the geometric case of [Zhang, 2001] (and [Amorós et al., 2000]), this (product) relation plays the role of the product formula for number fields.

Question 15. I don't know anything about tempered fundamental groups? Can one work with étale fundamental groups?

Ans. Tempered fundamental groups [André, 2003b], [André, 2003a] are pro-discrete groups and capture infinite discrete coverings such as that of Tate elliptic curves, while étale fundamental groups work with finite étale coverings. The tempered fundamental group requires one to work with Berkovich spaces. The datum of geometric base-points, for the tempered fundamental group, consists of fields which are algebraically closed and complete (rank-one) valued fields (and contain $\mathbb{Q}_p$ -isometrically). Tempered fundamental groups carry finer information and determine étale fundamental groups but not conversely. So one is necessarily working with a more refined object.

Question 16. I don't understand how working with geometric base-points changes anything!

Ans. The tempered or étale fundamental group is unaffected by the choice of the geometric base-point. But working with all geometric base-points is tantamount to working with an extended object. Let me explain this for étale fundamental groups. Say $X/\mathbb{C}$ is a complex quasi-projective variety. A geometric base-point for the étale fundamental group of X is a morphism of schemes $\text{Spec}(F) \rightarrow X$ for some algebraically closed field $F \supseteq \mathbb{C}$. For example, $F = \overline{\mathbb{C}((T))}$ is one such field. So working with all morphisms $*$: $\text{Spec}(\overline{\mathbb{C}((T))}) \rightarrow X$ is tantamount to working with $X(\overline{\mathbb{C}((T))})$. By the Newton-Pusieaux Theorem,

$$X(\overline{\mathbb{C}((T))}) = \bigcup_{n \geq 1} X(\mathbb{C}((T^{1/n}))).$$

Thus, this set contains $X(\mathbb{C}((T)))$ which is the algebraic loop-space of X , and it strictly contains $X(\mathbb{C})$ (in general).

Similarly, working with geometric base-points for tempered fundamental groups is like working with arithmetic loop spaces (this view is fleshed out in [Joshi, 2023a, § 9]). For relationship between working with all geometric base-points and with $X^\diamond$ of [Scholze, 2017, Section 15], see [Joshi, 2022, Remark 4.3], and with greater detail in [Joshi, 2021, Proposition 4.3.1].

Question 17. What is the valuation scaling in the above example?

Ans. In the above example, the valuation scaling alluded to earlier is this: if $|T|_{\mathbb{C}((T))} = c$ for some real number $0 < c < 1$, then

$$|T|_{\mathbb{C}((T^{1/n}))} = c^n.$$

Question 18. In the p -adic context, what is the valuation scaling analogous to Question 16, Question 17 and how does it relate to Mochizuki's assertions in [Mochizuki, 2021d, Theorem 3.11, Remark 3.11.1]?

Ans. In the context and notation of Question 8, if

$$|p|_{\mathbb{C}_p} = \frac{1}{p}$$

(this is the standard normalization), then

$$|p|_K = \frac{1}{p^\lambda}$$

for some $\lambda \in \mathbb{R}$. This is precisely the sort of relationship Mochizuki asserts in [Mochizuki, 2021d, Theorem 3.11 and Remark 3.11.1]. The non-trivial aspect here is that the valued field $(K, |\cdot|_K)$ is not isomorphic to $(\mathbb{C}_p, |\cdot|_{\mathbb{C}_p}^\nu)$ for any $\nu \in \mathbb{R}$ as the fields K are not topologically isomorphic. So this scaling of valuations given above arises from the fact that arithmetic holomorphic structures are distinct, and this scaling is quite non-trivial in general (as [Mochizuki, 2021d, Theorem 3.11] requires). Mochizuki sometimes refers to this sort of valuation scaling property as “*Teichmüller dilatations*.” On the other hand replacing $(\mathbb{C}_p, |\cdot|_{\mathbb{C}_p})$ by $(\mathbb{C}_p, |\cdot|_{\mathbb{C}_p}^\lambda)$ is a trivial scaling of valuation and that is not what one wants to do at all. In classical Teichmüller Theory, the existence of Teichmüller dilatations is encapsulated in the Beltrami differential or the Beltrami coefficient [Imayoshi and Taniguchi, 1992]. [One word of caution: the global Frobenius morphism (discussed in my answer to Question 24) also leads to valuation scaling and this aspect plays a central role in both Mochizuki’s and my approach to the establishment of the key inequalities of [Mochizuki, 2021d], [Mochizuki, 2021e] and [Joshi, 2024c], [Joshi, 2024b].]

Question 19. Mochizuki’s approach is supposed to be anabelian, so how does your approach reconcile with that?

Ans. Let me begin with the simple case of $\text{Spec}(\mathbb{Q}_p)$. The two fields $K, \mathbb{C}_p$ discussed earlier, provide two choices of geometric base-points $*_K : \text{Spec}(K) \rightarrow \text{Spec}(\mathbb{Q}_p)$ (resp. $*_{\mathbb{C}_p} : \text{Spec}(\mathbb{C}_p) \rightarrow \text{Spec}(\mathbb{Q}_p)$) (induced by the inclusions $\mathbb{Q}_p \hookrightarrow K$ and $\mathbb{Q}_p \hookrightarrow \mathbb{C}_p$) for computing the étale (resp. tempered) fundamental group of $\text{Spec}(\mathbb{Q}_p)$ (resp. $\text{Spec}(\mathbb{Q}_p)^{\text{an}}$). In this example, the étale and the tempered fundamental groups coincide. The (étale or tempered) fundamental group computed using the algebraic closure of $\mathbb{Q}_p$ in K (resp. $\mathbb{C}_p$) is $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)_K$ (resp. $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)_{\mathbb{C}_p}$). Now from elementary Galois theory one has a topological isomorphism

$$\begin{array}{ccc} \pi_1^{\text{ét}}(\text{Spec}(\mathbb{Q}_p), * : \text{Spec}(K) \rightarrow \text{Spec}(\mathbb{Q}_p)) & = & \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)_K \\ & & \Big| \simeq \\ \pi_1^{\text{ét}}(\text{Spec}(\mathbb{Q}_p), * : \text{Spec}(\mathbb{C}_p) \rightarrow \text{Spec}(\mathbb{Q}_p)) & = & \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)_{\mathbb{C}_p} \end{array}$$

between the fundamental groups computed using the respective geometric base-points. But there is no isomorphism of the pairs

$$(\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)_K, K) \text{ and } (\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)_{\mathbb{C}_p}, \mathbb{C}_p).$$

The latter is what *Mochizuki’s Key Principle of Inter-Universality* requires one to work with. In [Joshi, 2021] I prove that fundamental groups remain fixed (i.e. are isomorphic) under such arithmetic base changes, but the Berkovich analytic spaces are not isomorphic.

Question 20. So are you suggesting that working the pairs $(\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)_K, K)$ as above is adequate for this theory?

Ans. No, not at all. One should work with all untits $(K, K^b \simeq F)$ of all algebraically closed perfectoid fields F of characteristic p . In practice, it is enough to take $F = \mathbb{C}_p^b$. There are two reasons for this (1) the tilting data $K^b \simeq \mathbb{C}_p^b$ provides a fixed value group (namely that of $\mathbb{C}_p^b$) in which one can compare valuations of elements of K as the untit $(K, K^b \simeq \mathbb{C}_p^b)$ varies. For similar valuation comparisons, Mochizuki works with the pair consisting of an “*étale picture*” and a “*Frobenius picture*.” [This is detailed in [Joshi, 2024c].] (2) Working with the collection of untits means one has a Frobenius morphism (as in [Fargues and Fontaine, 2018]). [In [Joshi, 2024c] I provide a ‘Rosetta Stone’ for translating between my theory and Mochizuki’s theory.]

Question 21. But isn't all what you have said local for one prime number p ?

Ans. True. To go from one prime number to all prime numbers one uses adelic methods.

Question 22. But aren't adelic methods still local?

Ans. Yes and no. Adelic constructions are typically built using local objects and then one uses some global glue to glue all the local pieces together. For example, the product formula for a number field is one such global glue which glues all the local pieces together. This is the strategy employed by Mochizuki and myself in our papers. Apart from the product formula which we both use (for the same reason), there are two more global glues which we both use: (1) the global action of the multiplicative group of a number field in my work and Mochizuki's work (this action transforms arithmetic holomorphic structures keeping certain global volumes i.e. measures fixed), and (2) the action of a global Frobenius (called the log-Link in Mochizuki's work).

Question 23. Why have you asserted that Mochizuki's paper is incomplete?

Ans. My assertion of incompleteness is based on the genuine difficulty one encounters in using Mochizuki's toolkit in proving his assertions. The central difficulty is to show that there exist distinct arithmetic holomorphic structures (the analogous classical assertion is that the (isomorphic) elliptic curves E_τ and $E_{-\frac{1}{\tau}}$ represent distinct points of the classical Teichmüller space in genus one). [This is discussed in Question 12.] The second difficulty is to demonstrate the existence and non-triviality of various symmetries (Galois action, Θ -Links, log-Links/global Frobenius) asserted by Mochizuki. I prove these properties from an intrinsic geometric point of view in my papers.

Question 24. How can a Number Field have a global Frobenius morphism? I don't understand what role does the global Frobenius morphism play in the proof?

Ans. First let me be clear: the global Frobenius morphism is a continuous mapping on the Arithmetic Teichmüller Space $\mathcal{Y}_L$ of topologically distinct avatars of arithmetic of L constructed in [Joshi, 2023a]. This mapping is a precise version of the Log-Link considered in [Mochizuki, 2021b, Page 32]. This global Frobenius morphism changes the arithmetic avatar of the number field at all primes p . To explain this global Frobenius morphism, let L be a number field. The global Frobenius arises from the (genuine) Frobenius morphism of a Fargues-Fontaine curve $\mathcal{Y}_{\mathbb{C}_p^\flat, L_p}$ at all primes $\mathfrak{p}|p$ and all prime numbers p (one must also include archimedean primes). [So my construction of the global Frobenius morphism is quite geometric.]

The global Frobenius morphism also appears in the geometric case of [Zhang, 2001]—this is detailed in [Joshi, 2023a, Appendix]. Now to the role of the global Frobenius morphism in the arithmetic case. At this point it is essential to know the vector bundle theoretic proof of geometric Szpiro Inequality given in [Kim, 2002]. The inequalities proved there are very similar to the bundle theoretic inequalities proved using the genuine Frobenius morphism for vector bundles proved in [Joshi and Pauly, 2015]. This global Frobenius morphism $\varphi_L : \mathcal{Y}_L \rightarrow \mathcal{Y}_L$ does a number of things in the arithmetic case:

- (1) it alters valuations at all finite primes of L by a p^{th} -power for all prime numbers p simultaneously (see [Joshi, 2024c] for details);
- (2) it allows us to speak of a number field L twisted by global Frobenius $L^{(1)}$ (just as one does in characteristic $p > 0$ and especially in [Joshi and Pauly, 2015]),

- (3) and compare arithmetic quantities on L with those on $L^{(1)}$. Mochizuki employs his log-Link to achieve this.
- (4) This use of global Frobenius/log-Link is very similar to the comparison of a vector bundle on a curve and its Frobenius twisted version in [Joshi and Pauly, 2015].

Question 25. Are you asserting that there is a global Frobenius morphism in the geometric case proved in [Zhang, 2001] (and [Amorós et al., 2000])?

Ans. Yes. There is a global Frobenius morphism in these geometric proofs, and it plays a central role in the proof of the geometric Szpiro inequality given in both [Zhang, 2001] and [Amorós et al., 2000]. [This explicit identification of the global Frobenius morphism and other relevant structures in their proofs was not made by these authors, but is detailed in [Joshi, 2023a, Appendix].] That is why I have said that these geometric proofs must be thoroughly understood to understand what Mochizuki and I assert.

Question 26. Doesn't the main theorem of [Mochizuki, 2015, Theorem 1.9] contradict [Mochizuki, 2021b] because that theorem asserts that a smooth, hyperbolic p -adic curve over a number field is determined uniquely (up to isomorphism) by its tempered fundamental group?

Ans. No. [Mochizuki, 2015, Theorem 1.9] does not determine the geometric base-point information, namely the algebraically closed, complete valued field at all, while *Mochizuki's Key Principle of Inter-Universality* [Mochizuki, 2021b, §13, Pages 25–26] (see Question 31) requires one to remember the geometric base-point information. So the said theorem sheds no light on the crucial ingredient of [Mochizuki, 2021b]. [The geometric analogy is this: the fundamental group of Riemann surface does not determine its holomorphic structure. Likewise, [Mochizuki, 2015, Theorem 1.9] does not determine the arithmetic holomorphic structure.]

Question 27. Can one prove Szpiro's conjecture for one elliptic curve by these methods?

Ans. Szpiro's Conjecture is about all elliptic curves over a fixed number field, just as the abc -conjecture is about all (primitive) (a, b, c) triples. Strictly speaking, there is no statement for a single elliptic curve or for a single abc -triple.

Question 28. Can one prove Szpiro's Conjecture directly by these methods?

Ans. That has yet to be demonstrated. Mochizuki uses the reduction to Vojta's Inequality for compactly bounded subsets (of $(\mathbb{P}^1 - \{0, 1, \infty\})(\bar{\mathbb{Q}})$) which he had established in [Mochizuki, 2010]. So that is the case Mochizuki establishes in his fourth paper (and I follow his proof in my fourth paper with some additional clarification and supply necessary missing details). It was shown in [van Frankenhuysen, 2002] that the abc -conjecture is equivalent to Vojta's Inequality and it was previously established by Paul Vojta that the relevant case of his inequality implies the Szpiro Conjecture.

Question 29. But didn't Mochizuki assert in April 2024 that you must be wrong?

Ans. Mochizuki's report on my work makes it clear that he had not read my global approach because no mention of [Joshi, 2023a] occurs in his report (he does not separate his global approach into a separate paper as I do); and secondly, as I have said in the current version (Feb 2025) of [Joshi, 2024b], in the first version (March 2024), I had missed the usage of the global Frobenius at a critical juncture. [This omission has been fixed in the current versions of my relevant papers on the arxiv.]

Question 30. The recent article [Boyd, 2025] discusses $\in$ -loops and (mathematical) universes in the context of [Mochizuki, 2021b]. How do they figure in your theory?

Ans. First and foremost: there is no reference to any $\in$ -loops in [Mochizuki, 2021a], [Mochizuki, 2021c] and therefore these do not occur in any proofs given therein.

- (1) In [Mochizuki, 2021d, Remark 3.11.1, Page 581–585], which appears *after* the proof of Theorem 3.11, Mochizuki reminds us that identifying all the arithmetic holomorphic structures leads to a contradiction. This is the first tacit and only reference to $\in$ -loops in the context of the proof of [Mochizuki, 2021b, Theorem 3.11, Page 573]. To understand this, it is useful to understand the geometric case using the notation of Question 12. If one works in the category of complex manifolds, then asserting that E_τ and $E_{-\frac{1}{\tau}}$ are distinct Riemann surfaces leads to a contradiction (as these are biholomorphic). However, the genus one Teichmüller space can be identified with complex upper half-plane and in this space $E_\tau, E_{-\frac{1}{\tau}}$ correspond to two distinct points $\tau, -\frac{1}{\tau}$. Hence, one can assert that they are distinct (in the genus one Teichmüller space) without encountering a contradiction. *This* is the picture Mochizuki wants to emulate in the arithmetic case.

However, Mochizuki does not construct Arithmetic Teichmüller Spaces (but I do) nor does he provide any way of distinguishing between arithmetic holomorphic structures, so Mochizuki reminds us in [Mochizuki, 2021d, Remark 3.11.1(i), Page 581–585] that the proofs of [Mochizuki, 2021d, Theorem 3.11 and Corollary 3.12] requires one to work with distinct arithmetic holomorphic structures (by assigning arbitrary labels $\dagger$ and $\ddagger$ if necessary). The gist of the said remark is that working with distinct arithmetic holomorphic structures is absolutely necessary, as one wants to average over a collection of such structures, to avoid logical contradictions or $\in$ -loops. Asserting that there is only one arithmetic holomorphic structure is just as incorrect as asserting that there is only one holomorphic structure on any oriented, compact surface of genus $g \geq 1$, and just as incorrect as asserting that all quasi-conformal mappings between Riemann surfaces are conformal mappings. I had considerable difficulty with Mochizuki’s assertion, but I discovered that Mochizuki’s claim of distinct arithmetic holomorphic structures (i.e. of distinct versions of arithmetic) *can* be made quite robust using [Scholze, 2012], [Fargues and Fontaine, 2018] without necessitating any meta-mathematical invocations. This is detailed in [Joshi, 2021], [Joshi, 2023a]. My approach provides a canonical way to disambiguate Mochizuki’s claims about arithmetic holomorphic structures and avoids any discussion of $\in$ -loops or any meta-mathematical considerations.

- (2) Finally, *after* the proof of the *abc*-conjecture is completed, there is a meta-mathematical discussion involving universes and $\in$ -loops in [Mochizuki, 2021e, § 3, Pages 701–721].

There is no other occurrence of universes or $\in$ -loops in [Mochizuki, 2021b, Pages 40–700]. So it is safe to say that $\in$ -loops, or any other meta-mathematical theorems, are not directly used in the actual proofs of the assertions in [Mochizuki, 2021b].

Question 31. But your answer above does not discuss universes which I asked about in my previous question? So what about that aspect?

Ans. True. Let me discuss this now. Here is what Mochizuki says ([Mochizuki, 2021b, Page 26]):

It is this fundamental aspect of the theory of the present series of papers – i.e.,
of relating the distinct set-theoretic universes associated to the distinct fiber func-
tors/basepoints on either side of such a non-ring/scheme- theoretic filter
 – that we refer to as **inter-universal**.

(this is Mochizuki’s Key Principle of Inter-universality which forms the core strategy for his work).

So Mochizuki’s view is that each geometric base-point be treated as belonging to a distinct universe. I think this is an interesting idea, *but* Mochizuki does not actually use universes in the main body of the proof [Mochizuki, 2021b, Pages 40–700] (also see Question 39). To use the above principle, geometric base-points would need to be tracked, but he actually does not track geometric base-points, and then, on several occasions (e.g. [Mochizuki, 2021b, Page 580, and Pages 445, 446, 451, and Remark 1.4.2 Page 453]) he reminds us that (a) there are distinct arithmetic holomorphic structures involved and (b) that we should also remember that various operations of his theory reference the ring structures provided by geometric base-points.

Note however, the tacit idea for many people, is that geometric base-points should refer to the field $\bar{\mathbb{Q}}_p$ (or $\mathbb{C}_p$), and as I observed early on in my work, working with $\mathbb{C}_p$ is certainly inadequate according to the above quote (and the theory of tempered fundamental groups [André, 2003b]).

My approach, on the other hand, and my definition [Joshi, 2021, Definition 4.1.1] of arithmetic holomorphic structure does track geometric base-points explicitly (the relationship between this definition and $X^\diamond$ of [Scholze, 2017] is in [Joshi, 2021, § 4.3]). My central observation is that my definition of arithmetic holomorphic structures raises the theory to the level of classical Teichmüller Theory, and is completely in line with Mochizuki’s ideas and in fact, it is a quantitatively precise version of his ideas. Moreover, the key point of the proof requires working with distinct arithmetic holomorphic structures (i.e. distinct universes according to Mochizuki’s own words and his assertions in [Boyd, 2025]). Apart from the fact that I do not agree with the assertion that all the needed meta-mathematics is carefully worked out in [Mochizuki, 2021b], my point is that one can in fact avoid this meta-mathematical conversation by carefully establishing the existence of distinct arithmetic holomorphic structures whose absence the meta-mathematics is invoked to placate.

Question 32. Can you spell out what Mochizuki’s Key Principle of Inter-Universality means in terms of Classical Teichmüller Theory?

Ans. The key distinction between Teichmüller Space in genus g and moduli space in genus g is the use of quasi-conformal mappings in the former and the use of holomorphic mappings in the latter. Now suppose we did not know about quasi-conformal mappings, but wanted to create some way of distinguishing isomorphic Riemann surfaces (e.g. E_τ and $E_{-\frac{1}{\bar{\tau}}}$) while remaining in the holomorphic category. Mochizuki suggests this can be achieved by using distinct universes labeled by distinct base-points for fundamental groups. The problem which Mochizuki faced in his work in the arithmetic case was this: how to think of non-scheme theoretic mappings (resp. quasi-conformal mappings in the geometric case) while operating entirely in terms of schemes (resp. holomorphic mappings). Mochizuki suggests the use of universes in this context.

My approach, on the other hand, abandons working with schemes and works with Berkovich rigid analytic spaces and arrives at a more intrinsic p -adic notion, paralleling the classical theory, possessing the key property (“Teichmüller dilatations”) of quasi-conformal mappings, and keeping Mochizuki’s idea of geometric base-points as labels intact.

Question 33. Can you explain this better, say, in the context of Question 16, Question 17?

Ans. Certainly. First note that, for each $n \geq 1$, one has an isomorphism of discretely valued fields $\sigma_n : \mathbb{C}((T)) \rightarrow \mathbb{C}((T^{1/n}))$ via the $\mathbb{C}$ -algebra isomorphism given by $T \mapsto T^{1/n}$. Note that on one hand $\mathbb{C}((T)) \subset \overline{\mathbb{C}((T))}$ and $\mathbb{C}((T^{1/n})) \subset \overline{\mathbb{C}((T))}$, but on the other hand, $\mathbb{C}((T))$ and $\mathbb{C}((T^{1/n}))$ are not isomorphic as subfields of $\overline{\mathbb{C}((T))}$.

Now suppose $\mathcal{E}/\mathbb{C}((T))$ is an elliptic curve with semistable reduction modulo T (for e.g. $\mathcal{E} : y^2 = x(x-1)(x-T)$). Now let $\mathcal{E}_n$ be the pull-back elliptic curve $\mathcal{E}/\mathbb{C}((T))$ via σ_n to an elliptic curve over $\mathbb{C}((T^{1/n}))$. Thus, one has isomorphic elliptic curves over isomorphic fields: $\mathcal{E}/\mathbb{C}((T)) \xrightarrow{\sigma_n} \mathcal{E}_n/\mathbb{C}((T^{1/n}))$. But one may also view $\mathcal{E}$ and $\mathcal{E}_n$ as curves over $\overline{\mathbb{C}((T))}$ and observe that they are not isomorphic over $\overline{\mathbb{C}((T))}$ (seen by computing their j -invariants). In Mochizuki’s parlance, one has distinct universes labeled by $\mathbb{C}((T))$ and $\mathbb{C}((T^{1/n}))$ respectively, and one has constructed a universe labeled by $\overline{\mathbb{C}((T))}$ in which the isomorphic elliptic curves $\mathcal{E}$ and $\mathcal{E}_n$ are not isomorphic. [Using the language of universes here is, of course, quite unnecessary once one has added the required precision to make such assertions.]

Question 34. I agree that this is correct, but how is this related to the question of bounds and inequalities?

Ans. I continue in the context and the notation of my answer to Question 33. Suppose Ψ is a set of non-isotrivial elliptic curves over $\mathbb{C}((T))$. The problem one has is how to compute an upper bound on a suitably defined volume of Ψ . By applying the above procedure to each $\mathcal{E} \in \Psi$ and each $n \geq 1$, one obtains a subset Ψ_n of elliptic curves over $\mathbb{C}((T^{1/n}))$, and a subset $\Psi' = \bigcup_{n \geq 1} \Psi_n$ of elliptic curves over $\overline{\mathbb{C}((T))}$, and obviously $\Psi' \supset \Psi_1 = \Psi$. Now suppose that, by some other method, one has obtained an upper bound for the volume of Ψ' , then one has (indirectly) solved the problem of bounding the volume of Ψ . The problem of estimating the volume of Ψ directly is difficult, but, estimating the volume of Ψ' may be easier. That is the sort of strategy Mochizuki uses in his paper.

Question 35. In your answer to Question 31 you mentioned the connection with $X^\diamond$ constructed in [Scholze, 2017, Section 15]. Can you say what the relationship is and what it means?

Ans. Let $X/\mathbb{Q}_p$ be a geometrically connected, smooth hyperbolic curve (e.g. the affine curve $y^2 = x(x-1)(x-p)$). A point of the local Arithmetic Teichmüller Space [Joshi, 2021, Definition 4.1.1] of $X/\mathbb{Q}_p$ consists of the following data:

$$\mathfrak{sol}_y(X) = (\overbrace{(\mathbb{Q}_p \hookrightarrow K_y, K_y^\flat \simeq \mathbb{C}_p^\flat)}^{\text{an untitl of } \mathbb{C}_p^\flat}, \overbrace{*_{K_y} : \mathcal{M}(K_y) \rightarrow X_{\mathbb{Q}_p}^{an}}^{\text{geometric base-point}}).$$

I call this datum a *holomorphoid* of $X/\mathbb{Q}_p$ (so this is at one prime p , but there is a global version in [Joshi, 2023a] and [Joshi, 2024c]). In this datum, one thinks of the untitl datum $(\mathbb{Q}_p \hookrightarrow K_y, K_y^\flat \simeq \mathbb{C}_p^\flat)$ given by $\mathfrak{sol}_y(X)$ as arising from a closed classical point of the Fargues-Fontaine curve $y \in \mathcal{Y}_{\mathbb{C}_p^\flat, \mathbb{Q}_p}$, with residue field K_y ; $\mathcal{M}(K_y)$ is the Berkovich spectrum of the Banach field K_y , and $*_{K_y} : \mathcal{M}(K_y) \rightarrow X_{\mathbb{Q}_p}^{an}$ is a morphism of Berkovich spaces.

As is pointed out in [Joshi, 2022, Remark 2.3], [Joshi, 2021, Proposition 4.3.1], $\mathfrak{H}\text{ol}_y(X)$ also arises (via Huber's functor to Berkovich spaces) from some

$$x = ((\mathbb{Q}_p \hookrightarrow K_y, K_y^\flat \simeq \mathbb{C}_p^\flat), *_{K_y} : \text{Spa}(K_y) \rightarrow X_{\mathbb{Q}_p}^{\text{ad}}) \in X^\diamond(\mathbb{C}_p^\flat),$$

where $X^\diamond$ is the diamond of $X/\mathbb{Q}_p$ constructed in [Scholze, 2017]. Note that a $\mathfrak{H}\text{ol}_y(X)$ explicitly keeps a track of a geometric base-point for $X_{\mathbb{Q}_p}^{\text{an}}$ and the tempered fundamental group gives a functor from the category of $\mathfrak{H}\text{ol}_y(X)$ to the category of pro-discrete groups given by

$$\mathfrak{H}\text{ol}_y(X) \longmapsto \Pi_y = \pi_1^{\text{temp}}(X_{\mathbb{Q}_p}^{\text{an}}, *_{K_y} : \mathcal{M}(K_y) \rightarrow X_{\mathbb{Q}_p}^{\text{an}}).$$

So $\mathfrak{H}\text{ol}_y(X)$ is a very natural object which exists for intrinsic reason and there is nothing artificial about it and whose existence, and its relationship with Mochizuki's Theory (which works with the group Π_y) or with $X^\diamond$ of [Scholze, 2017, Section 15], is mathematically robust and cannot be (mathematically) denied. In the archimedean case i.e. $p = \infty$, one may think of a point of the classical Teichmüller space as a holomorphoid of a Riemann surface. The global arithmetic versions of holomorphoids appear in [Joshi, 2023a], [Joshi, 2024c]. In my work, I show that a quantitatively precise version of Mochizuki's Theory is obtained by working with holomorphoids (local and global) of elliptic curves (my work is much more general). The proof of the *abc*-conjecture which Mochizuki and I are asserting rests on comparing and averaging over arithmetic holomorphic structures on elliptic curves i.e. on holomorphoids of elliptic curves.

Question 36. Now how do Question 33 and Question 34 play out in the arithmetic case?

Ans. This is spelled out in great detail in my papers [Joshi, 2021], [Joshi, 2023a] using the language of arithmeticoids and arithmetic holomorphic structures and leads to the construction of the analog of Ψ' in the arithmetic context. *Here is a very rough idea.* Suppose $\Theta = \{\mathbf{q}_E = \{q_v\}_{v \in \mathbb{V}_L} : E/L\}$ is the set of Tate parameters of some chosen subset elliptic curves over L . [In [Joshi, 2024a] and [Mochizuki, 2021e], this a compactly bounded subset of j -invariants.] We want to estimate suitably defined size of Θ . *This is difficult.* Mochizuki's idea (in my formulation) is to carefully chose a subset $S \subset \mathcal{Y}_L$ of arithmeticoids of L . Consider $E_{;\mathbf{y}}$ which is E as an elliptic curve with respect to the arithmeticoid $\mathbf{y} \in S$, more precisely, $E_{;\mathbf{y}} = \mathfrak{H}\text{ol}_{\mathbf{y}}(E)$ is a (global) holomorphoid of E with $\mathbf{y} \in S$, and enlarge $\Theta \subset \Theta'$ by taking

$$\Theta' = \{\mathbf{q}_{E_{;\mathbf{y}}} = \{q_{E_{;\mathbf{y}},v}\}_{v \in \mathbb{V}_L} : E_{;\mathbf{y}} \text{ where } \mathbf{y} \in S\}$$

and estimate the size of Θ' instead. Mochizuki's observation is that if the subset S is chosen carefully to be stable under various symmetries of $\mathcal{Y}_L$, then estimating the volume of Θ' in a suitable adelic space is (delicate but) actually doable. *Strictly speaking*, S is not a subset of $\mathcal{Y}_L$, but of a larger space. Details can be found in [Joshi, 2024c].

Question 37. But if the versions of arithmetic $\mathbf{y} \in S$ are supposed to be different how can the Tate parameters $\mathbf{q}_{E_{;\mathbf{y}}}$ arising from different versions of arithmetic given by $\mathbf{y}$ be compared?

Ans. This is an important point and is best understood by first understanding the situation at one prime p . There are two approaches to this. One using Galois cohomology (this was pioneered by Mochizuki) and a second approach, which I discovered, using the rings $A_{\text{inf},p} = W(\mathcal{O}_{\mathbb{C}_p^\flat}) \subset B_p$ considered in [Fargues and Fontaine, 2018]. Both the approaches are essentially equivalent (via the relationship between $A_{\text{inf},p}$, B_p and Galois cohomology). My approach is conceptually cleaner, while Mochizuki's is better suited for computations. This is detailed

in [Joshi, 2023b], [Joshi, 2024c], but let me illustrate the key ideas. I will use the notation considered in Question 35). Say $X/\mathbb{Q}_p$ is an elliptic curve with semi-stable reduction at p . Let $\mathfrak{H}_{\text{ol}_{y_1}}(X)$ and $\mathfrak{H}_{\text{ol}_{y_2}}(X)$ be two holomorphoids of this curve and for $i = 1, 2$, let $q_i \in K_{y_i}^*$ be the Tate parameters of X computed in the fields K_{y_1} and K_{y_2} respectively. Let $A_{\text{inf},p} = W(\mathcal{O}_{\mathbb{C}_p^\flat})$, from [Fargues and Fontaine, 2018], one knows that y_1, y_2 correspond to principle prime ideals $\mathfrak{p}_{y_i} \subset A_{\text{inf},p}$ and one has a surjective homomorphism $A_{\text{inf},p} \rightarrow A_{\text{inf},p}/\mathfrak{p}_{y_i} \simeq \mathcal{O}_{K_{y_i}} \ni q_i$. By [Fargues and Fontaine, 2018, Chapter 2], one has Teichmüller lifts $x_i \in A_{\text{inf},p}$ of q_i . The lifts are not canonical, but one has a certain natural collection of lifts, and one considers such lifts. This is detailed in [Joshi, 2023b]. Mochizuki's approach via Galois cohomology rests on the facts that y_1, y_2 give isomorphic Galois cohomologies:

$$q_1 \in H^1(\text{Gal}(\bar{\mathbb{Q}}_p)_{y_1}, \mathbb{Q}_p(1)_{y_1}) \simeq H^1(\text{Gal}(\bar{\mathbb{Q}}_p)_{y_2}, \mathbb{Q}_p(1)_{y_2}),$$

and the Tate parameters give rise to Galois cohomology classes in these groups. There is no canonical isomorphism between these groups, and in general, q_1 is not carried into q_2 under such isomorphisms. Hence, one must consider all $\mathbb{Q}_p$ -vector space isomorphisms between these groups arising from isomorphisms of the pairs $(\text{Gal}(\bar{\mathbb{Q}}_p)_{y_1} \curvearrowright \mathbb{Q}_p(1)_{y_1}) \simeq (\text{Gal}(\bar{\mathbb{Q}}_p)_{y_2} \curvearrowright \mathbb{Q}_p(1)_{y_2})$ and collate images of Tate parameters q_1 by fixing one y_2 (usually chosen in some natural way). The general global setup is somewhat complicated and adelic with the product formula and a global action of the number field playing a key role in both, Mochizuki's papers and my papers. This is detailed in [Joshi, 2024c].

Question 38. The article [Boyd, 2025] also mentions the log-Theta lattice in [Mochizuki, 2021b]?

Ans. This is best understood using the geometric case first. I gave its construction in the geometric case in [Joshi, 2023a, Appendix]. The set S considered in the previous answer uses Mochizuki's Theta-Link which underlies his log-Theta lattice.

Question 39. But according to [Boyd, 2025] and [Mochizuki, 2021b, Page 701], Mochizuki refers to models of set theory in the context of his proof. Is this true?

Ans. The short answer is that Mochizuki's Key Principle of Inter-Universality referred to in Question 31, does lead to the existence of distinct models of set theories as Mochizuki asserts. Mochizuki does not provide a proof of this, but this can be proved using the properties of the classifying topos of the tempered fundamental groups (namely, this topos is an atomic, Boolean, Grothendieck topos). *However*, let me again emphasize that such models are not used in [Mochizuki, 2021b, Pages 40–700] i.e. in the main body of the proof of the *abc*-conjecture, and the focus on universes and models of set theories in [Boyd, 2025] is misleading.

Question 40. But neither [Boyd, 2025] nor Mochizuki refer to your work. Why is that?

Ans. This is not a question for me. But the lack of citation to my work is indeed quite troubling.

Question 41. But what about [Scholze and Stix, 2018] and [Scholze, 2021]?

Ans. This question is answered in [Joshi, 2025] (and [Joshi, June, 2024]).

Question 42. What can one do with this theory?

Ans. The theory is still nascent, and I am not prescient enough to predict all its consequences just as no one could have foreseen the rich mathematical tapestry that classical Teichmüller Theory has come to be (see [Papadopoulos, 2007]) by just looking at Teichmüller’s paper in the 1940s.

In 2018, during my sabbatical visit to RIMS (Kyoto) hosted by Mochizuki, I told Mochizuki that his methods make the higher genus case of Szpiro’s conjecture ([von Kānel, 2013]) tractable. The higher genus geometric case is established by vector bundle theoretic methods in [Kim, 2002], [Beauville, 2002] and Teichmüller theoretic methods in [Bogomolov, Katzarkov, and Pantev, 2002].

In [Joshi, 2019], I have provided an algebraization of Mochizuki’s monoidal point of view and established its relationship with the monoidal aspect of the theory of perfectoid rings. This monoidal aspect is also related to the approach (via monoid rings) in [Deninger, 2025] (which uses the ring of rational Witt vectors, while [Joshi, 2019] uses the ring of Witt vectors). Recently, on reading [Sédillot, 2024], I became aware of the work [Chen and Moriwaki, 2020] on complete adelic curves. Here is how this relates to my work. Each arithmeticoid i.e. a point $y \in \mathcal{Y}_L$ of [Joshi, 2023a] naturally provides a complete adelic curve in the sense of [Chen and Moriwaki, 2020, Example 3.2.2]. So that $\mathcal{Y}_L$ is also a Teichmüller space for complete adelic curves arising from a number field. On the other hand, from [Chambert-Loir, 2025, Example 1.1.3] one sees that each arithmeticoid $y \in \mathcal{Y}_L$ also provides a global valued field in the sense of [Yaacov, Destic, Hrushovski, and Szachniewicz, 2024] and hence one may think of $\mathcal{Y}_L$ as a Teichmüller space for global valued fields arising from the number field L . Two important features of my work are (1) the existence of global Frobenius morphism on this Arithmetic Teichmüller Space $\mathcal{Y}_L$, and (2) the construction of the *product-formula-as-a-period* morphism $\mathcal{Y}_L \rightarrow \mathbb{P}(\bigoplus_{v \in M_L} \mathbb{R}_v)$ where M_L is the set of places of L and $\mathbb{R}_v = \mathbb{R}$ for each $v \in M_L$, and this morphism associates to each $y \in \mathcal{Y}_L$ the hyperplane $H_y \subset \mathbb{P}(\bigoplus_{v \in M_L} \mathbb{R}_v)$ given by the product formula for y , established in [Joshi, 2023a, Theorem 5.10.1(7)]. The product formula is also central to the global considerations in [Chen and Moriwaki, 2020], [Yaacov et al., 2024]. So I expect that other applications of Arithmetic Teichmüller Spaces to Diophantine geometry to emerge eventually.

Appendix

Here are some additional pointers regarding the arithmetic case discussed in Questions 11–14. It will be useful to first fix a prime number p and understand the simplest case of $X = \text{Spec}(\mathbb{Q}_p)$. If one takes two copies $X_1 = \text{Spec}(\mathbb{Q}_p)$ and $X_2 = \text{Spec}(\mathbb{Q}_p)$, there is no way of distinguishing them in the language of schemes as $X_1 \simeq X_2$ as $\mathbb{Q}_p$ schemes. Let $\mathbb{C}_p$ be the completion of some fixed algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$; let $\mathbb{C}_p^b$ be its tilt [Scholze, 2012]. The path which I found in my work to resolve this is as follows: let $\mathcal{Y}_{\mathbb{C}_p^b, \mathbb{Q}_p}$ be the Fargues-Fontaine curve constructed in [Fargues and Fontaine, 2018] using input datum $(\mathbb{C}_p^b, \mathbb{Q}_p)$. Closed classical points of this curve parameterize untilts of $\mathbb{C}_p^b$ i.e. pairs $(\mathbb{Q}_p \hookrightarrow K, K^b \simeq \mathbb{C}_p^b)$ where K is an algebraically closed perfectoid field, $\mathbb{Q}_p \hookrightarrow K$ is an isometric embedding, and $K^b \simeq \mathbb{C}_p^b$ is a given isometry between K^b and $\mathbb{C}_p^b$. One should think of the untilt datum $(\mathbb{Q}_p \hookrightarrow K, K^b \simeq \mathbb{C}_p^b)$ as the p -adic analog of an arithmetic holomorphic structure or a p -adic analog of a Riemann surface equipped with a quasi-conformal mapping (precise definitions are given in [Joshi, 2021]). Given a pair of distinct points $y_1, y_2 \in \mathcal{Y}_{\mathbb{C}_p^b, \mathbb{Q}_p}$, by forgetting the additional untilt data, one obtains schemes

$X_1 = \text{Spec}(\mathbb{Q}_p)$ and $X_2 = \text{Spec}(\mathbb{Q}_p)$. Therefore, in the language of untilts, one may solve the problem of distinguishing X_1, X_2 by working instead with y_1, y_2 , just as one does in Classical Teichmüller Theory.

How does this relate to Mochizuki’s Key Principle of Inter-Universality quoted in the answer to Question 31? Here is how: each $y_1 = (\mathbb{Q}_p \hookrightarrow K, K^\flat \simeq \mathbb{C}_p^\flat)$ provides the algebraically closed, perfectoid field K needed for computing one fiber functor for the tempered fundamental groups (and Galois groups) which Mochizuki refers to in his key principle. Notably, in my view, $\mathcal{Y}_{\mathbb{C}_p^\flat, \mathbb{Q}_p}$ is the analog of the classical Teichmüller space of $X = \text{Spec}(\mathbb{Q}_p)$.

In [Joshi, 2021], [Joshi, 2022], [Joshi, 2023a], I show that this approach extends to quasi-projective varieties over p -adic fields and number fields, provides a satisfactory resolution of the key issues, keeping with Mochizuki’s Key Principle. Notably in [Joshi, 2023a], for each number field L (assumed to have no real embeddings for simplicity—this is also assumed by Mochizuki), I construct an Arithmetic Teichmüller Space $\mathcal{Y}_L$, whose points $y \in \mathcal{Y}_L$ may be thought of as $y = (\text{number field } L \text{ plus a Teichmüller datum for } L)$, and in which one can assert $y_1 \neq y_2$ (for $y_1, y_2 \in \mathcal{Y}_L$) even though both correspond to fixed number field L . In the parlance of [Mochizuki, 2021b, Remark 3.11.1, Page 580], y_1 and y_2 represent two distinct arithmetic holomorphic structures. The existence of $\mathcal{Y}_L$ (together with its list of properties, such as the global Frobenius morphism $\varphi : \mathcal{Y}_L \rightarrow \mathcal{Y}_L$ which is a precise version of Mochizuki’s log-Link) is the minimal requirement for formulating Mochizuki’s assertions (e.g. [Mochizuki, 2021b, Theorem 3.11, Page 573]) without self contradictions. The assertions of [Scholze and Stix, 2018] arose from the fact that [Mochizuki, 2021b] does not provide adequate mathematical language for asserting the existence of distinct arithmetic holomorphic structures i.e. of asserting $y_1 \neq y_2$, and I agree with them on this; however they asserted, incorrectly, that it was impossible to find distinct arithmetic holomorphic structures, and I demonstrate the existence of arithmetic holomorphic structures in my papers.

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