Math 129 Section 005H
Lecture 39: Taylor series solutions to differential equations (Ch. 11 supplement)

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Note: Here is a PDF version of this file.

Reminders

- WA 11.1 due Monday 11/22
- Written HW 12 due Monday 11/22
- Yellowdig
- Tutoring

Taylor series and differential equations

Today I covered variations of Examples 3(b) and 5 from the Ch. 11 Supplement. We end with this:

Theorem: Taylor series solutions of differential equations If $p(x), q(x), r(x)$ have convergent Taylor series in an interval $-R < x < R$ for $R > 0$, then the solution to the *initial value problems*[1]

$$\frac{dy}{dx} + p(x)y = q(x)\ , \quad y(0) = a$$

and

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)\ , \quad y(0) = a\ , \quad y'(0) = b$$

[1]The phrase "initial value problem" refers to a differential equation together with enough initial conditions to uniquely determine a solution, i.e., a first-order initial value problem means a first-order differential equation with one initial condition, a second-order initial value problem means a second-order differential equation with an initial value and an initial derivative, etc.

can be expressed as a Taylor series that also converges on $-R < x < R$:

$$y(x) = C - 0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots .$$

Alternative method

An alternate (but mathematically equivalent) method is by repeated differentiation. This is illustrated by the following example.

Suppose we want to solve

$$y''(x) + y'(x) + y(x) = 0 , \quad y(0) = 1, \quad y'(0) = 2.$$

As usual, we write

$$y(x) = C_0 + C_1 x + C_2 x^2 + \cdots .$$

Now: if we just plug $x = 0$ into the differential equation, we get

$$y''(0) + y'(0) + y(0) = 0.$$

With the initial conditions, this means we have

$$y(0) = 1, y'(0) = 2, y''(0) = -y'(0) - y(0) = -3.$$

From this, we already know

$$y(x) = 1 + 2x - \frac{3}{2!} x^2 + \cdots .$$

To get the next term in the Taylor series, we can differentiate the differential equation:

$$\frac{d}{dx}\big(y''(x) + y'(x) + y(x)\big) = y'''(x) + y''(x) + y'(x) = 0.$$

Plugging in $x = 0$, we get

$$y'''(0) + y''(0) + y'(0) = 0$$

so that $y'''(0) = -y''(0) - y'(0) = -3 - 2 = -5$ and

$$y(x) = 1 + 2x - \frac{3}{2} x^2 - \frac{5}{3!} x^3 + \cdots ,$$

and so on. Which method to use is a matter of preference, and also depends a bit on the form of the differential equation – this method is simpler when repeated differentiation is easy to perform.