Homework 6 additional problem

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1) For this problem, it is useful to know that two events A and B are *conditionally independent* given a third event C if

$$P(A \cap B|C) = P(A|C) \cdot P(B|C). \tag{1}$$

   (a) Show that if $X_0, X_1, \cdots, X_n, \cdots$ is a Markov chain, then for every $m, n \geqslant 0$ and $x_i \in S$,

$$P\Big(\underbrace{X_0 = x_0, X_1 = x_1, \cdots, X_{n-1} = x_{n-1}}_{\text{past}}\ ,\ \underbrace{X_{n+1} = x_{n+1}, \cdots, X_{n+m} = x_{n+m}}_{\text{future}}\ \Big|\ \underbrace{X_n = x_n}_{\text{now}}\Big)$$

$$= P\Big(\underbrace{X_0 = x_0, X_1 = x_1, \cdots, X_{n-1} = x_{n-1}}_{\text{past}}\ \Big|\ \underbrace{X_n = x_n}_{\text{now}}\Big) \cdot P\Big(\underbrace{X_{n+1} = x_{n+1}, \cdots, X_{n+m} = x_{n+m}}_{\text{future}}\ \Big|\ \underbrace{X_n = x_n}_{\text{now}}\Big)$$

   In words: for a Markov chain, the future and the past are conditionally independent given the present.[1]

   (b) Suppose a chain X_n is irreducible. Show that for all $n \geqslant 0$ andstates x, y, z with $y \neq x$, the events $(T_x > n)$ and $(X_{n+1} = z)$ are conditionally independent given $X_n = y$.

2) Let P be a stochastic matrix with the given eigenvalues.[2] Say whether the Markov chain described by P is (i) ~~irreducible~~ has a unique stationary distribution, and (ii) converges to equilibrium. Briefly justify.

   (a) $\left\{\pm 1, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}, -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right\}$

   (b) $\left\{1, 1, \pm\frac{1}{2}\right\}$

   (c) $\left\{\pm 1, \pm\frac{1}{3}\right\}$

   (d) $\left\{1, -\frac{1}{2}, -\frac{1}{2}\right\}$

3) Let P be an $N \times N$ doubly stochastic matrix, and suppose the corresponding Markov chain (X_n) is irreducible. Show that (X_n) satisfies the detailed balance condition if and only if P is a symmetric matrix.

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[1] The converse is actually also true, but I'm not asking you to show that here.

[2] A repeated eigenvalue has the indicated multiplicity, i.e., "1,1" means 1 is a double root of the characteristic equation $\det(P - \lambda I) = 0$.