YOUR NAME: KEY

Math 129-005H Homework 4 (improper integrals)[1]

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Here is a PDF version of this document.

1) Does

$$\int_0^\infty e^{-t}\,dt$$

converge or diverge? If it converges, find its value.

Solution:

$$\begin{aligned}\int_0^b e^{-t}\,dt &= \left[-e^{-t}\right]\Big|_{t=0}^{b} \\ &= -e^{-b}+e^{0} \\ &= 1-e^{-b}.\end{aligned}$$

Since $e^{-b}\to 0$ as $b\to\infty$, the integral converges to $\boxed{1}$.

2) Does

$$\int_0^\infty te^{-t}\,dt$$

converge or diverge? If it converges, find its value.

Solution: integrate by parts with $u = t$ and $v' = e^{-t}$:

$$\begin{aligned}\int_0^b te^{-t}\,dt &= \left[-te^{-t}\right]\Big|_{t=0}^{b} + \int_0^b e^{-t}\,dt \\ &= -be^{-b} + 0 + 1 - e^{-b}\end{aligned}$$

Since $be^{-b} \to 0$ and $e^{-b} \to 0$ as $b \to \infty$, the integral converges to $\boxed{1}$.

3) The *gamma function* is defined for all $x \geqslant 1$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

It appears widely in many areas of mathematics and physics, for example in probability theory. (The function is actually defined for all $x > 0$, but for $x < 1$ it involves another kind of improper integral which we will discuss next week.)

a. What are the values of $\Gamma(1)$ and $\Gamma(2)$? *Hint: you've already done all the work above! No need to redo the work.*

Solution: $\boxed{\Gamma(1) = \int_0^\infty t^{1-1} \cdot e^{-t}\, dt = 1}$ and $\boxed{\Gamma(2) = \int_0^\infty t^{2-1} \cdot e^{-t}\, dt = 1}$

b. *Start with the definition of* $\Gamma(x+1)$ *and integrate by parts* once *with respect to* t *to find a simple relation between* $\Gamma(x+1)$ *and* $\Gamma(x)$*. This will lead to an expression for*

$$\frac{\Gamma(x+1)}{\Gamma(x)};$$

what is it? (This problem may be a little confusing because there's both x *and* t*. You should think of* x *as a constant and treat* t *as the variable.)*

Solution:

$$\begin{aligned} \Gamma(x+1 &= \int_0^\infty t^{x+1-1} e^{-t}\, dt \\ &= \int_0^\infty t^x e^{-t}\, dt \end{aligned}$$

Integrate by parts with $u = t^x$, $v' = e^{-t}$:

$$\begin{aligned} \int_0^b t^x e^{-t}\, dt &= -t^x e^{-t}\Big|_{t=0}^{b} + \int_0^b x t^{x-1} e^{-t}\, dt \\ &= -b^x e^{-b} + 0 + \int_0^b x t^{x-1} e^{-t}\, dt \end{aligned}$$

if $x > 0$. As $b \to \infty$, $b^x e^{-b} \to 0$, so

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty x t^{x-1} e^{-t}\, dt \\ &= x \int_0^\infty t^{x-1} e^{-t}\, dt \qquad \text{(because } x \text{ is a "constant" here)} \\ &= x\Gamma(x). \end{aligned}$$

Thus $\boxed{\Gamma(x+1)/\Gamma(x) = x}$.

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c. *Using the above, find a simple expression for* $\Gamma(n)$. Hint: try calculating $\Gamma(3)$ in terms of $\Gamma(2)$, $\Gamma(4)$ in terms of $\Gamma(3)$, etc. What pattern do you see?

Solution:

$$\begin{aligned}
\Gamma(1 &= 1 \\
\Gamma(2 &= 1 \\
\Gamma(3 &= \Gamma(2+1) \\
&= 2 \cdot \Gamma(2) \\
&= 2 \cdot 1 \\
\Gamma(4 &= \Gamma(3+1) \\
&= 3 \cdot \Gamma(3) \\
&= 3 \cdot 2 \cdot 1
\end{aligned}$$

From this pattern one can see that $\boxed{\Gamma(n) = (n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 = (n-1)!.}$