Computing the Cohomology Ring and Ext-Algebra of Group Algebras

by

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Table of Contents

Table of Contents— *Continued*

Table of Contents— *Continued*

List of Figures

List of Tables

Abstract

This dissertation describes an algorithm and its implementation in the computer algebra system GAP for constructing the cohomology ring and Ext-algebra for certain group algebras $\Bbbk G$. We compute in the Morita equivalent basic algebra B of $\Bbbk G$ and obtain the cohomology ring and Ext-algebra for the group algebra $\Bbbk G$ up to isomorphism. As this work is from a computational point of view, we consider the cohomology ring and Ext-algebra via projective resolutions.

There are two main methods for computing projective resolutions. One method uses linear algebra and the other method uses noncommutative Gröbner basis theory. Both methods are implemented in GAP and results in terms of timings are given. To use the noncommutative Gröbner basis theory, we have implemented and designed an alternative algorithm to the Buchberger algorithm when given a finite dimensional algebra in terms of a basis consisting of monomials in the generators of the algebra and action of generators on the basis.

The group algebras we are mainly concerned with here are for simple groups in characteristic dividing the order of the group. We have computed the Ext-algebra and cohomology ring for a variety of simple groups to a given degree and have thus added many more examples to the few that have thus far been computed.

Introduction

To study a finite group, one could study the representations of that group. A representation ρ of a finite group G over a field $\Bbbk$ is a homomorphism $\rho : G \to \mathrm{GL}(V)$ of G into the group $\mathrm{GL}(V)$ of invertible $\Bbbk$-endomorphisms of a finite dimensional vector space V over $\Bbbk$. Another way of studying a finite group is to study the structure of a related ring. We do this by constructing a finite dimensional vector space with G as a basis and defining a suitable multiplication. We call this ring $\Bbbk G$ the group algebra. Any representation $\rho : G \to \mathrm{GL}(V)$ of a finite group G over a field $\Bbbk$ (and likewise any matrix representation) extends by $\Bbbk$-linearity naturally to a ring homomorphism $\rho : \Bbbk G \to \mathrm{End}(V)$ which will be denoted by the same symbol and which is called a representation of the group ring. Also V becomes a $\Bbbk G$-module with $v \cdot a := v\rho(a)$ for $a \in \Bbbk G$ and $v \in V$. Conversely, if V is any $\Bbbk G$-module which has finite dimension as a $\Bbbk$-vector space, then one obtains a representation $\rho : \Bbbk G \to \mathrm{End}(V)$ by defining $v\rho(a) := v \cdot a$ and one obtains a representation of G by restricting ρ to G. The $\Bbbk G$-module V is often called the representation module of the representation $\rho : G \to \mathrm{GL}(V)$. Obviously equivalent representations have representation modules which are isomorphic as $\Bbbk G$-modules and vice versa. Thus to study a group one can study the representations of the group or one can study the $\Bbbk G$-modules. We will take the point of view in this dissertation of studying the $\Bbbk G$-modules.

Thus we may think of finitely generated $\Bbbk G$-modules as being the same thing as representations of G as matrices with entries in $\Bbbk$. Ideally, we would like to classify all $\Bbbk G$-modules for a group G and a field $\Bbbk$. However, the field $\Bbbk$ plays an important role. According to Maschke's theorem if the characteristic p of the field $\Bbbk$ does not divide the order of G, then we know that all $\Bbbk G$-modules are the direct sum of simple modules. When p does divide the order of G, this is no longer true. In this situation, a new class of interesting modules arises which are no longer the direct sum of simple modules. However, any finitely generated $\Bbbk G$-module still has a composition series. The reconstruction of a $\Bbbk G$-module in the case where p divides the order of the group G from simple composition factors is far more complicated. This is a highly nontrivial task which we call the extension problem. A useful approach to attacking the extension problem is applying methods from homological algebra.

As a starting point in homological algebra we consider a $\Bbbk G$-module M with simple submodule S_1 and quotient module $S_2 = M/S_1$. Then M can be expressed as the following exact sequence:

$$0 \longrightarrow S_1 \longrightarrow M \longrightarrow S_2 \longrightarrow 0.$$

All such extensions modulo a suitable equivalence relation form a $\Bbbk$-vector space $\mathrm{Ext}^1_{\Bbbk G}(S_2, S_1)$. If we consider longer exact sequences starting in S_1 and ending in S_2 of a fixed length $n \geq 2$, we may similarly define higher $\mathrm{Ext}^n_{\Bbbk G}(S_2, S_1)$. These higher Ext-vector spaces are needed for reconstructing modules with composition series of length n. Therefore, we are interested in determining $\mathrm{Ext}^n_{\Bbbk G}(S_i, S_j)$ for all simple $\Bbbk G$-modules. In order to get a grip on all of these vector spaces $\mathrm{Ext}^n_{kG}(S_i, S_j)$, we note that for sequences in $\mathrm{Ext}^n_{\Bbbk G}(S_i, S_j)$ and sequences in $\mathrm{Ext}^m_{\Bbbk G}(S_j, S_k)$ we can splice these sequences together to get an element of $\mathrm{Ext}^{m+n}_{\Bbbk G}(S_i, S_k)$. In this way, we can consider $\dot{+}_{i,j,n}\, \mathrm{Ext}^n_{\Bbbk G}(S_i, S_j)$ which is not only a $\Bbbk$-vector space but also a graded $\Bbbk$-algebra. This algebra is known as the Ext-algebra. Although we have an infinite dimensional vector space, Evens [Eve61] has shown that the Ext-algebra is finitely generated as a $\Bbbk$-algebra. Therefore, one goal is to describe this noncommutative infinite dimensional algebra in terms of a finite set of generators and the relations satisfied among the generators.

The definition of an Ext-algebra in terms of equivalence classes of long exact sequences is a useful theoretical tool. However, for computational purposes a more practical way of describing the Ext-algebra is by using minimal projective resolutions [CGS97]. The literature covers two generally different ways of carrying out this computation; for example see [CTVEZ03] and [Gre97]. A minimal projective resolution for a simple module M may be defined in an iterative way: We take an epimorphism ε from a projective module $P(M)$ of minimal dimension mapping onto M. We call the kernel of this map $\Omega^1(M)$. We then compute an epimorphism from $P(\Omega^1(M))$ onto $\Omega^1(M)$, take the kernel of this map and continue. We can summarize this in the following sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & P(\Omega^2(M)) \arrow[rr, "\partial_2"] \arrow[dr, "\omega_2"'] & & P(\Omega^1(M)) \arrow[rr, "\partial_1"] \arrow[dr, "\omega_1"'] & & P(M) \arrow[r, "\varepsilon"] & M \\
& & \Omega^2(M) \arrow[ur, "\iota_2"'] & & \Omega^1(M) \arrow[ur, "\iota_1"'] & &
\end{tikzcd}$$

As mentioned above, there are two different ways to compute a minimal projective resolution. For the first approach to this problem we consider the exact sequence as a sequence of linear maps between finite dimensional vector spaces and use basic ideas from linear algebra to compute the resolution. The second approach is referred to as the Anick-Green resolution [Gre99]. The idea is to represent our homomorphisms in a much more compact way than as large matrices with entries in $\Bbbk$, i.e. linear maps. This is accomplished via noncommutative Gröbner basis theory. The idea is to work with maps between projective modules as lists of generator images. I have implemented both of these techniques in the computer algebra system GAP [GAP05].

All finite simple groups have been classified and we would like to better understand them through various methods such as computing Ext-algebras. We first reduce the amount of work we have to do by studying an equivalent algebra B, called a basic algebra, which has much smaller dimension as a $\Bbbk$-vector space. The fact that allows us to do this is that a group algebra $\Bbbk G$ and its equivalent basic algebra B have isomorphic Ext-algebras.

We now have access to a database of basic algebras for some large groups and the ability to compute more [Hof04]. We are supplied with a faithful representation of the basic algebra in terms of matrices. The data that is given is already fit for the linear algebra techniques of computing the Ext-algebra. However, to use the Anick-Green technique we need a presentation of the algebra in terms of generators and relations where the generators for the relations ideal are given as a Gröbner basis. We have implemented an algorithm in GAP that gives a Gröbner basis presentation for basic algebras. It is an alternative to the noncommutative version of the Buchberger algorithm. This has allowed us to give an efficient Gröbner basis presentation for the basic algebra of large simple groups such as the Higman Sims group in characteristic 2.

Historically, the algorithms of noncommutative Gröbner bases and computation of projective resolutions have been implemented by Green and Feustel in a C-program called GRB [FG91]. Unfortunately, this program is restricted to the base field $\mathbb{F}_p$ and does not work for larger fields $\mathbb{F}_{p^n}$. We naturally have to consider extensions of $\mathbb{F}_p$ to study even small groups such as the alternating group A_5 in characteristic 2. We have implemented this algorithm over arbitrary finite fields in GAP. Coming back to the linear algebra approach, J. Carlson [CTVEZ03] has implemented this technique in the computer algebra system MAGMA [MAG04]. His work focuses mainly on p-groups in characteristic p. In that setting, all group algebras are already basic and the Ext-algebra is the same as the cohomology ring. Carlson has computed the cohomology ring of 2-groups up to order 128. In the case of p-groups there is only the trivial simple module and thus in some ways is an easier problem. We thus aim to study examples of arbitrary larger simple groups. We have completed the implementation for computing the cohomology ring and Ext-algebra of a group algebra in GAP. We present the Ext-algebra and cohomology ring as the quotient of a path algebra.

One important question in computing an Ext-algebra is when have we found all of the generators and sufficiently many relations. However, this is an extremely difficult question. There are two types of results that give specific criteria that guarantee a sufficient set of generators and relations have been found. Benson and Carlson [BC87] give such a criterion which can be applied to a special situation. The other type of result exploits the structure of algebras of a special type such as algebras of dihedral type. This case includes groups with dihedral Sylow subgroups (see [Gen01, GO02, Gen02, GK03, GK04]). The theoretical base for this problem needs to be expanded.

In general, not much is known about Ext-algebras. Benson and Carlson have made Ext-algebra computations using diagrammatic methods [BC87]. In that paper they computed $\mathrm{Ext}^*_{\Bbbk G}(S, S)$ for all simple modules of $\mathbb{F}_2 M_{11}$, $\mathbb{F}_3 A_6$, $\mathbb{F}_2 L_3(3)$, $\mathbb{F}_2 A_7$, $\mathbb{F}_2 S_4$, and $\mathbb{F}_2 D_8$. Carlson has also calculated $H^*(G, \mathbb{F}_2) = \mathrm{Ext}^*_{\mathbb{F}_2 G}(\mathbb{F}_2, \mathbb{F}_2)$ for 2-groups up to order 128 in the computer algebra system MAGMA [MAG04]. With the implementations of our programs in GAP, we shall be able to build a large library of Ext-algebras for more groups and look for new results.

In the first chapter we give the basic results from ring, module, and algebra theory. The first step is to study the group algebra $\Bbbk G$. However as we would like to study some rather large groups such as the sporadic simple Higman Sims group which has size $44, 352, 000$ we would like to study a smaller object which shares the same properties as our original object. Thus we will use the Morita equivalent basic algebra B which is categorically equivalent to our original algebra $\Bbbk G$. We will discuss the basics of Morita theory in section 1.6.2. The reduction of the size of the algebra can be quite significant. For example, the basic algebra of $\Bbbk G$, where G is the sporadic simple group Higman Sims, for $\Bbbk$ a field of characteristic 2, has dimension $2, 462$ over $\Bbbk$. This is now an algebra that can efficiently be worked with on a computer.

The second step is to construct the projective resolution for all of the simple $\Bbbk G$-modules. The projective resolutions of the simple $\Bbbk G$-modules are the same as the projective resolutions of the simple B-modules and so we will work with the basic algebra B. The problem that has to be solved is: given a map $\partial_n : P_n \to P_{n-1}$ of projective modules, construct a map $\partial_{n+1} : P_{n+1} \to P_n$ which is the projective cover of the kernel of the map ∂_n. In practice, the way we represent ∂_n on the computer has an important effect on performance. Thus we will investigate and compare the two different approaches to this problem by implementing these procedures in GAP.

The linear algebra approach is to represent ∂_n by its matrix as a map of $\Bbbk$-vector spaces. Constructing a basis for the kernel of ∂_n then involves taking the null space of the matrix. The only data needed about the basic algebra are the matrices for the action of the generators in the regular representation. Using these matrices, we obtain vectors spanning the radical of the kernel of the map ∂_n, and then use linear algebra and some theory about finite dimensional algebras to find a basis for the complement of the radical. Then P_{n+1} has one projective summand for each basis vector and we can take ∂_{n+1} to map the generator of the i^{th} summand to the i^{th} basis vector. We discuss the theory and implementation of this approach in chapter 2 and the implementation in chapter 4.

As noted, we are interested in studying groups that can be extremely large. Therefore the linear algebra approach has the limitation of memory storage due to the storage of rather large matrices. However, this approach is efficient in speed as linear algebra over finite fields can be done rather quickly in GAP with standard commands. Thus we would like to have a method that has a more efficient storage method. That is we would like to be able to represent our homomorphisms in a much more compact way than as a large matrix with entries in $\Bbbk$, i.e. linear maps. The idea is to use noncommutative Gröbner basis theory and to work with maps between projective modules as lists of generator images. And as we have a Gröbner basis theory we will have a unique normal form that we can work with. Using noncommutative Gröbner bases we can manipulate modules and maps sorted as finite presentations and lists of generator images respectively, although with some redundancy in the presentations. The theory is built upon the theory of noncommutative Gröbner bases that arise from quotients of path algebras. We will outline the theory of Gröbner bases and the corresponding method and implementation of computing projective resolutions in chapter 3.

Once we have computed the minimal projective resolutions, we wish to compute the cohomology ring and the Ext-algebra for a group algebra $\Bbbk G$. The last algorithm we implement in this thesis is a procedure for computing the Ext-algebra and cohomology ring of a group algebra up to a given degree. The most important feature of our program will be to have an effective way of lifting homomorphisms and computing chain maps. We describe the theory of Ext-algebras, cohomology rings, and how to compute them in chapter 2. We describe the implementation in chapter 4. Ultimately, we present our algebra abstractly in terms of generators and relations, where the relations ideal I is given as a Gröbner basis $\mathcal{G}$.

We end the dissertation with some of the computational results that we were able to obtain using our implementations in GAP. We give these in chapter 5. We end chapter 5 with some concluding remarks about the two different implementations we have made in GAP for projective resolutions and give a sample of timing comparisons for various groups for the linear algebra approach in GAP, the Gröbner basis approach in GAP, and the program GRB.

Chapter 1

Background

In this chapter we first present a background of basic terminology and results from ring, algebra, and module theory. We then go on to provide the necessary results that are needed for our algorithms and implementation in GAP. For a background on the basics of rings, algebras, and modules good references can be found in Grove [Gro04] and Dummitt and Foote [DF91]. For more advanced topics we refer the reader to Curtis and Reiner [CR90], Auslander [ARS95], Benson [Ben98a, Ben98b], and Carlson [Car96]. Most of the results in this chapter are well-known. We will, however, include parts of the proof or the whole proof where we have found appropriate, for example when we have not found a thorough or good proof in the literature or when we want to emphasize a point.

1.1 Rings, Algebras and Modules

Definition 1.1.1. *A ring A with identity 1_A is said to be an* ***algebra*** *over a commutative ring R, or an R-algebra if there exists a homomorphism $\psi : R \to Z(A)$ from R into the center $Z(A)$ of A, such that $\psi(1_R) = 1_A$.*

One of the main goals of this thesis is from a representation theorist's point of view and thus we will be interested in studying the group algebra of a finite group. Throughout the dissertation we assume that G is a finite group and $\Bbbk$ is a field of positive characteristic p unless otherwise noted.

Definition 1.1.2. *If G is a group, $\Bbbk$ a field, then the* ***group algebra*** *$\Bbbk G$ is the set of all formal finite sums*

$$\left\{ \sum_{x \in G} \alpha_x x : \alpha_x \in \Bbbk \right\},$$

i.e. a vector space with a basis of all group elements with addition defined as

$$\left(\sum_{x\in G}\alpha_x x\right)+\left(\sum_{x\in G}\beta_x x\right)=\sum_{x\in G}(\alpha_x+b_x)x.$$

We give $\Bbbk G$ *a ring structure by defining multiplication as*

$$\left(\sum_{x\in G}\alpha_x x\right)\left(\sum_{y\in G}\beta_y y\right)=\sum_{x,y\in G}\alpha_x\beta_y xy.$$

This group ring $\Bbbk G$ *is a* $\Bbbk$*-algebra by virtue of the embedding* $\Bbbk \to \Bbbk G$*, given by* $\alpha\cdot 1\to\alpha\cdot 1_G$*,* $\alpha\in\Bbbk$*, where* 1_G *is the identity in* G*.*

When studying algebras as in many objects in mathematics, we are interested in the corresponding sub-objects. Thus we define:

Definition 1.1.3. *Let* A *be an algebra with a subring* B*. If* B *is also a* $\Bbbk$*-vector subspace of* A*, we call* B *a* ***subalgebra*** *of* A*.*

As we study ideals in rings, we also study ideals in algebras.

Definition 1.1.4. *Let* A *be a* $\Bbbk$*-algebra. A* ***right ideal*** I *in the algebra* A *is a subalgebra of* A *which is also a right ideal in the ring* A*. Left ideals are defined similarly. If* I *is both a right and left ideal in* A*, then we call it a* ***two-sided ideal****.*

A main focus in this dissertation is algorithms for modules. To study an algebra A, we will be looking at finitely generated A-modules.

Definition 1.1.5. *Let* A *be an algebra over* $\Bbbk$*. We say that* M *is a* ***right*** A***-module*** *(resp. a left module) if it is a* $\Bbbk$*-vector space with a right action (resp. left action) by* A *satisfying:*

1. $m\cdot(a_1a_2)=(m\cdot a_1)\cdot a_2$,

2. $m\cdot(a_1+a_2)=m\cdot a_1+m\cdot a_2$,

3. $(m_1 + m_2) \cdot a = m_1 \cdot a + m_2 \cdot a$*,*

4. $m \cdot (1_A) = m$*,*

5. $\lambda (m \cdot a) = (\lambda m) \cdot a$*, for all* $m, m_1, m_2 \in M$ *and* $a, a_1, a_2 \in A$*.*

If M is a right A-module and a left B-module for two algebras A and B such that $b \cdot (m \cdot a) = (b \cdot m) \cdot a$ for all $b \in B$, $a \in A$, and $m \in M$, the we call M a B-A **bimodule**.

Throughout we will assume that all of our modules are finitely generated.

One example of a module that occurs quite often in representation theory is the following:

Definition 1.1.6. *The* ***right regular*** *A-module of A is given as follows: We allow A to act on itself as a right module by right multiplication and denote it by A_A. We can similarly define a left regular module by multiplication on the left.*

Definition 1.1.7. *A* ***representation*** *ρ of a group G over a field $\Bbbk$ is a homomorphism $\rho : G \to \mathrm{GL}(V)$ of G into the group $\mathrm{GL}(V)$ of invertible $\Bbbk$-endomorphisms of a finite n-dimensional vector space V over $\Bbbk$. We call n the* ***degree of the representation****.*

Example **1.1.1.** *Let A be a group algebra. The right regular module A_A is a representation. It is called the* ***right regular representation****.*

Example **1.1.2.** *Let G be a finite group of order 3, $G = C_3 = \langle a : a^3 = 1 \rangle$. Let $\Bbbk$ be any field. The elements of $\Bbbk G$ have the form*

$$\lambda_1 \cdot 1 + \lambda_2 \cdot a + \lambda_3 \cdot a^2 \qquad (\lambda_i \in \Bbbk)$$

We see that

$$\left(\lambda_1 \cdot 1_G + \lambda_2 \cdot a + \lambda_3 \cdot a^2\right) \cdot 1 = \lambda_1 \cdot 1 + \lambda_2 \cdot a + \lambda_3 \cdot a^2,$$
$$\left(\lambda_1 \cdot 1 + \lambda_2 \cdot a + \lambda_3 \cdot a^2\right) \cdot a = \lambda_3 \cdot 1 + \lambda_1 \cdot a + \lambda_2 \cdot a^2,$$
$$\left(\lambda_1 \cdot 1 + \lambda_2 \cdot a + \lambda_3 \cdot a^2\right) \cdot a^2 = \lambda_2 \cdot 1 + \lambda_3 \cdot a + \lambda_1 \cdot a^2.$$

By taking matrices relative to the basis 1*,* a*, and* a^2 *of* $\Bbbk G$*, we obtain the right regular representation of* G*:*

$$1 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, a \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, a^2 \to \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Note that the matrices for the regular representation are $n \times n$ where n is the order of the group G.

Definition 1.1.8. *Two representations* $\rho, \sigma : G \to \mathrm{GL}(V)$ *are said to be* ***equivalent*** *if there is an invertible homomorphism* ψ *such that for all* $g \in G$ *we have* $\rho(g) = \psi \cdot \sigma \cdot \psi^{-1}(g)$*.*

As we are most interested in group algebras, we will always have a trivial representation.

Definition 1.1.9. *The representation* $\rho : G \to \mathrm{GL}(V)$ *over a* 1*-dimensional vector space* V *defined by* $g\rho = 1$ *for all* $g \in G$ *is called the* ***trivial representation****. The* ***trivial*** $\Bbbk G$***-module*** *is the one dimensional vector space* V *with* $vg = v$ *for all* $v \in V$ *and* $g \in G$*.*

Definition 1.1.10. *Let* M *be an* A*-module and* N *a* $\Bbbk$*-vector subspace of* M*. Then* N *is called an* A***-submodule*** *of* M *if* $n \cdot a \in N$ *for all* $a \in A$ *and* $n \in N$*.*

Let I be a right ideal in the finite dimensional algebra A. Then I is also a right A-module. In fact, the right ideals of A are the submodules of A_A.

Example **1.1.3.** *Continuing example 1.1.2 from above, we see that if we let* $w = 1 + a + a^2$*, then* $W = \mathrm{Span}_{\Bbbk}(w)$ *is a submodule of the right regular module* $\Bbbk G_{\Bbbk G}$*.*

We give a special name to modules that have only trivial submodules. We will see later that these are building blocks of all finitely generated modules over a finite dimensional algebra.

Definition 1.1.11. *A* ***simple*** *A-module is a nonzero A-module S whose only submodules are 0 and S. Sometimes a simple module is also referred to as irreducible.*

A concrete way of considering whether or not a $\Bbbk G$-module M is simple is to consider the corresponding matrices of the representation. Suppose that the representation given is of degree n. We view M as a submodule of $\Bbbk^n$. Suppose that M is not a simple module, i.e. it is reducible. So there is a $\Bbbk G$-submodule N with $0 < \dim N < \dim M$. Take a basis $\mathcal{B}_1$ of N and extend it to a basis $\mathcal{B}$ of M. Then for all g in G, the matrix $[g]_{\mathcal{B}}$ has the form

$$\begin{bmatrix} A_g & 0 \\ B_g & C_g \end{bmatrix} \tag{1.1}$$

for some matrices A_g, B_g, and C_g where A_g is $m \times m$ ($m = \dim N$).

A representation of degree n is reducible if and only if it is equivalent to a representation of the form (1.1), where A_g is $m \times m$ and $0 < m < n$. Note that in (1.1), the homomorphisms $\rho : g \to A_g$ and $\psi : g \to C_g$ are representations of G. Thus from the above we know that a representation is irreducible if and only if it cannot be put into this form.

When studying modules, we also wish to study the maps between them. We will most often be interested in A-module homomorphisms between modules. Recall that the A-modules M and N are $\Bbbk$-vector spaces, so we can consider the $\Bbbk$-linear maps between M and N. We are most interested in the $\Bbbk$-linear maps which commute with the action of A on M and N. These maps are the A-homomorphisms.

Definition 1.1.12. *Let M and N be A-modules. Then a $\Bbbk$-linear map $\varphi : M \to N$ is an A-**homomorphism** if $\varphi(m \cdot a) = \varphi(m) \cdot a$ for all $m \in M$ and $a \in A$. We use $\mathrm{Hom}_A(M, N)$ to denote the $\Bbbk$-vector space of A-homomorphisms from M to N and $\mathrm{End}_A(M)$ to denote $\mathrm{Hom}_A(M, M)$. If φ is a bijection then it is called an A-**isomorphism**.*

One of the easiest theorems in representation theory that is often very useful is known as Schur's Lemma. We will use it later to help determine possible maps between simple modules.

Lemma 1.1. *(Schur's Lemma) If M is a simple A-module then $\mathrm{End}_A(M)$ is a division ring . If N is another simple A-module, then either M and N are isomorphic or else $\mathrm{Hom}_A(M, N) = 0$.*

Proof. A straightforward proof is found in Grove [Gro04, page 173]. $\square$

One important sequence of homomorphisms that is important to us in our constructions is the following.

Definition 1.1.13. *Let $M_1, ..., M_n$, be A-modules with homomorphisms $f_i : M_i \to M_{i+1}$ for $i = 1, ..., n-1$. If $\mathrm{Im}(f_i) = \mathrm{Ker}\, f_{i+1}$ then we call the sequence*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$$

***exact** at M_{i+1}. If it is exact at $M_2, ..., M_{n-1}$ then we say that the sequence is an **exact sequence**. If the sequence*

$$0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0$$

*is an exact sequence we call it a **short exact sequence**.*

In our later construction of the Anick-Green resolution, we are interested in exact sequences that split. By this we mean:

Definition 1.1.14. *A short exact sequence*

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0 \tag{1.2}$$

*is called **split** if there is an A-module homomorphism $g : M_3 \to M_2$ such that $f_2 \circ g$ is the identity map on M_3.*

A further interpretation of the definition of split exact is that in (1.2) above we would have that $M_2 \cong_A M_1 \oplus M_3$.

Another important concept in module theory is the notion of a largest submodule.

Definition 1.1.15. *By a **maximal submodule** of an A-module M we mean a submodule $N \subset M$ such that there are no submodules L with $N \subset L \subset M$.*

Note: We use the notation $\subset$ throughout and will always mean contained in but not equal.

We also can characterize maximal submodules in terms of kernel of epimorphisms. For each epimorphism $f : M \twoheadrightarrow S$ with S simple, we know that $\operatorname{Ker} f$ is a maximal submodule of M. Conversely, if L is maximal in M, then L is the kernel of the natural surjection $M \twoheadrightarrow M/L$. Thus, we can characterize maximal submodules of M as kernels of surjections $M \twoheadrightarrow S$, with S simple.

As well as simple modules, a main focus for our study will be indecomposable modules.

Definition 1.1.16. *An A-module M is called **indecomposable** if it cannot be written as a direct sum of two non-trivial submodules. It is called **decomposable** otherwise.*

Definition 1.1.17. *An A-module M is called **semisimple** if it is the direct sum of a family of simple submodules. A ring A is called **semisimple** if A_A is semisimple.*

According to the next theorem, we know that a semisimple algebra is a sum of simple algebras and the simple summands are isomorphic to matrix algebras.

Theorem 1.2. *(Wedderburn-Artin Structure Theorem) Let A be a semisimple algebra with r isomorphism classes of simple modules S_i, with $i = 1, ..., r$. Then A is an external direct sum of full matrix algebras, $A \cong \dot{+}_{i=1}^{r} \operatorname{Mat}_{n_i}(\Delta_i)$, where Δ_i is a division ring such that $\Delta_i \cong \operatorname{End}_A(S_i)$ and $n_i = \dim_{\Delta_i}(S_i)$.*

Proof. A proof of the Wedderburn Theorem is found in Benson [Ben98a, page 6]. □

We know from another theorem of Wedderburn that every finite division ring is a field. Therefore, in the case of a finite dimensional $\Bbbk$-algebra A, we have that Δ_i is a finite extension of $\Bbbk$.

Definition 1.1.18. *If A is an algebra over a field $\Bbbk$ and S is a simple A-module, then $\Bbbk$ is called a* ***splitting field*** *for S if $\operatorname{End}_A S = \Bbbk \cdot id_M$.*

Basically, a splitting field for an algebra A is a field $\Bbbk$ such that for all possible field extensions $\Bbbk'$, the simple A-modules remains simple over $\Bbbk'$.

The next theorem is one of the main dividing points between ordinary and modular representation theory.

Theorem 1.3. *(Maschke) If $\Bbbk$ is a field and G a finite group, then the group algebra $\Bbbk G$ is semisimple if and only if the characteristic of $\Bbbk$ is not a divisor of the group order $|G|$.*

Proof. For a proof see Grove [Gro04, page 176]. □

According to Maschke's theorem we know that if $p \nmid |G|$ then indecomposable and irreducible are the same. However, if $p \mid |G|$ then it is only true that irreducible implies indecomposable.

Example **1.1.4.** *The conclusion of Maschke's theorem can fail if $\Bbbk$ is not $\mathbb{R}$ or $\mathbb{C}$, that is the characteristic of the field $\Bbbk$ divides the order of the group, $p \mid |G|$. For* *example let p be a prime number, let $G = C_p = \langle a : a^p = 1\rangle$, the cyclic group of order p, and take $\Bbbk$ to be the field of integers modulo p, $\mathbb{F}_p$. The operation*

$$a^j \to \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} \ (j = 0, 1, ..., p-1)$$

is a representation from G to $\mathrm{GL}(2, k)$. The corresponding $\Bbbk G$-module is the $\Bbbk$-linear span $M = \mathrm{Span}_{\Bbbk}(v_1, v_2)$, where, for $0 \leq j \leq p-1$,

$$v_1 a^j = v_1 + j v_2,$$

$$v_2 a^j = v_2.$$

Thus, $N = \mathrm{Span}_{\Bbbk}(v_2)$ is a $\Bbbk G$-submodule of M. But there is no $\Bbbk G$-submodule N' such that $M = N \oplus N'$, since N is the only 1-dimensional $\Bbbk G$-submodule of M.

As in the explanation of an irreducible $\Bbbk G$-module of $\Bbbk^n$ in terms of matrices, we can similarly define what it means to be indecomposable. A module is decomposable if and only if there is a change of basis such that the matrices of our representation can be put into the form

$$\begin{bmatrix} A_g & 0 \\ 0 & B_g \end{bmatrix}. \tag{1.3}$$

Thus a representation is indecomposable if and only if no change of basis can be found to put it into the above form (1.3).

Another type of module that we need to discuss that will be of importance to us is a generalization of a free module F. The easiest example of a free module is a vector space.

Definition 1.1.19. *An A-module P is said to be* ***projective*** *if given A-modules M and N, a map $\lambda : P \to N$ and an epimorphism $\mu : M \twoheadrightarrow N$ there exists a map $f : P \to M$ such that the following commutes:*

$$\begin{tikzcd} & P \arrow[dl, "\exists f"'] \arrow[d, "\lambda"] \\ M \arrow[r, two heads, "\mu"] & N \end{tikzcd}$$

The above definition of a projective module is not the only way to think of a projective module. The following proposition gives us other ways we can consider projective modules.

Proposition 1.4. *Let P be any A-module where A is a $\Bbbk$-algebra. Then the following are equivalent.*

1. *P is a projective module*
2. *P is a direct summand of a free module*
3. *Every epimorphism $\lambda : M \to P$ splits.*

Proof. See Dummitt and Foote[DF91, page 375]. $\square$

***Example* 1.1.5.** *Consider the $\Bbbk$-algebra of all 2×2 matrices over $\Bbbk$ and denote it by $\mathcal{M}$. Then we can take $\mathcal{M}$ as a right $\mathcal{M}$ module with an action of right multiplication. This is a free module. Consider the projective module*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

This is a projective module as it is a direct summand of the free module above. However, it is not free as

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

As modules that are both projective and indecomposable are important for us, we make a definition.

Definition 1.1.20. *An A-module P that is both projective and indecomposable is called a **projective indecomposable module** which we refer to as a **PIM**.*

1.2 Radicals and Socles

In the study of A-modules, we will often look at a specific submodule that is important in our constructions.

Definition 1.2.1. *The* ***radical*** *of an A-module M (denoted by $\operatorname{Rad} M$) is defined as the intersection of all maximal submodules of M. (If M has no maximal submodules, set $\operatorname{Rad} M = M$.)*

For every nonzero finitely generated A-module M, we can also characterize the radical in terms of homomorphisms from M to simple modules as follows:

$$\operatorname{Rad} M = \bigcap \operatorname{Ker} f,$$

where the intersection is taken over all epimorphisms $f : M \twoheadrightarrow S$, with S simple.

Example **1.2.1.** *If M is a simple $\Bbbk G$-module, then $\operatorname{Rad} M = 0$ since 0 is the only maximal submodule of M. More generally, $\operatorname{Rad} M = 0$ for every semisimple right $\Bbbk G$-module M. Note, however that it may well happen in a more general setting that $\operatorname{Rad} M = 0$ even though M is not a semisimple module. For example, let $M = \mathbb{Z}$, a right $\mathbb{Z}$-module. The maximal submodules of M are given by $\{p\mathbb{Z} : p \text{ is prime}\}$, and their intersection is 0. Thus $\operatorname{Rad} M = 0$, but M cannot be expressed as a direct sum of simple submodules.*

Definition 1.2.2. *The* ***radical series*** *or* ***Loewy series*** *of M is defined inductively by $\operatorname{Rad}^0(M) = M$, $\operatorname{Rad}^n(M) = \operatorname{Rad}\left(\operatorname{Rad}^{n-1}(M)\right)$ and the n^{th}* ***radical layer*** *or* ***Loewy layer*** *is $\operatorname{Rad}^{n-1}(M) / \operatorname{Rad}^n(M)$.*

In representation theory another important submodule is the following.

Definition 1.2.3. *The* ***socle*** *of an A-module M is the sum of all the irreducible submodules of M, denoted $\operatorname{Soc}(M)$.*

Note: We may also define a module M is to be semisimple (completely reducible) if $M = \operatorname{Soc}(M)$.

Definition 1.2.4. *The **head** or **top** of a module M is*

$$\operatorname{Head}(M) := M/\operatorname{Rad}(M)\,.$$

As the radical plays an important role for us in our constructions later, we are interested in some basic properties of radicals.

Proposition 1.5. *Let N, M be A-modules.*

1. *For each A-homomorphism $g : N \to M$, we have $g(\operatorname{Rad} N) \subseteq \operatorname{Rad} M$.*
2. *If $N \subseteq M$, then $\operatorname{Rad} N \subseteq \operatorname{Rad} M$, and $(\operatorname{Rad} M + N)/N \subseteq \operatorname{Rad}(M/N)$.*
3. *If $N \subseteq \operatorname{Rad} M$, then $(\operatorname{Rad} M)/N = \operatorname{Rad}(M/N)$.*

Proof. For a proof see Proposition 5.1 in Curtis and Reiner [CR90, page 103]. □

We immediately get a useful corollary from Proposition 1.5.

Corollary 1.6. *Let M be an A-module. Then $M/\operatorname{Rad} M$ has radical 0 and RadM is the smallest submodule M' of M such that $\operatorname{Rad}(M/M') = 0$.*

Proof. By Proposition 1.5.3, with $N = \operatorname{Rad} M$, we have

$$\operatorname{Rad}(M/(\operatorname{Rad} M)) = (\operatorname{Rad} M)/(\operatorname{Rad} M) = 0.$$

Conversely, if $\operatorname{Rad}(M/M') = 0$, then by Proposition 1.5 it follows that $\operatorname{Rad} M \subseteq M'$. □

As we have defined the radical of a module, similarly we define the notion of a radical for a $\Bbbk$-algebra A.

Definition 1.2.5. *The **Jacobson radical of** A, denoted by* $\operatorname{Jac} A$*, is the radical of the right regular module* A_A*. Thus*

$$\operatorname{Jac} A = \bigcap M, M \text{ ranging over all maximal right ideals of } A.$$

We have a proposition for radicals of rings similar to proposition 1.5 for modules.

Proposition 1.7. *Let* A *be a* $\Bbbk$*-algebra.*

1. *The factor ring* $A/\operatorname{Jac} A$ *has radical* 0*.*

2. *For any algebra epimorphisms* $f : A \twoheadrightarrow B$*, we have* $f(\operatorname{Jac} A) \subseteq \operatorname{Jac} B$ *and* f *induces an epimorphism* $A/\operatorname{Jac} A \twoheadrightarrow B/\operatorname{Jac} B$*.*

3. *For each right* A*-module* M*,* $M \cdot \operatorname{Jac} A \subseteq \operatorname{Rad} M$*.*

Proof. For a proof see proposition 5.6 in Curtis and Reiner [CR90, page 105]. □

In our construction of an Ext-algebra $E(A)$ we will begin with a finite dimensional algebra A and end up with $E(A)$ which is infinite dimensional. However, it has a grading to it. We thus make the following definitions.

Definition 1.2.6. *A **graded vector space** is a vector space* V *which can be written as a direct sum of the form*

$$V = \bigoplus_{n \in \mathbb{N}} V_n$$

where each V_n *is a finite dimensional vector space. For a given* n *the elements of* V_n *are then called **homogeneous elements of degree** n.*

Graded vector spaces are common. For example the set of all polynomials in one variable form a graded vector space, where the homogeneous elements of degree n are exactly the polynomials of degree n.

Definition 1.2.7. *A* ***graded algebra*** *A is an algebra that has a direct sum decomposition as a graded vector space*

$$A = \bigoplus_{i \in \mathbb{N}} A_i = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

such that

$$A_m \cdot A_n \subseteq A_{m+n}$$

Elements of A_n are known as homogeneous elements of degree n.

Since rings may be regarded as $\mathbb{Z}$-algebras, a graded ring is defined to be a graded $\mathbb{Z}$-algebra.

Examples of graded algebras are common in mathematics:

***Example* 1.2.2.** *The most common example of a graded algebra is a polynomial ring. The homogeneous elements of degree n are exactly the homogeneous polynomials of degree n.*

Definition 1.2.8. *The corresponding idea in module theory is that of a* ***graded module****, namely a module M over a graded algebra A such that*

$$M = \bigoplus_{i \in \mathbb{N}} M_i,$$

as a graded vector space and

$$M_j \cdot A_i \subseteq M_{i+j}.$$

1.3 Noetherian and Artinian Rings

Throughout this work our motivation is in dealing with studying finite dimensional algebras and their modules. Thus we would like to characterize radicals of rings and the corresponding finitely generated modules in this specific situation. Finite dimensional algebras are part of a more general class of rings known as Artinian and Noetherian rings. When dealing with these specific type of rings, we have some other characterizations of the radical of a ring. So we shall first discuss Artinian rings and Noetherian rings and then present some of their properties.

Definition 1.3.1. *A right A-module M is said to be* ***Noetherian*** *if the submodules of M satisfy the ascending chain condition (ACC), i.e., for every increasing sequence of submodules of M,*

$$M_1 \subseteq M_2 \subseteq \cdots,$$

there exists an integer n such that $M_n = M_{n+1} = \cdots$.

Definition 1.3.2. *The ring A is said to be right* ***Noetherian*** *if A_A is Noetherian, i.e., if there are no increasing chains of right ideals in A.*

Proposition 1.8. *Any finite dimensional algebra A is Noetherian. Thus the group algebra $\Bbbk G$ of a finite group G is Noetherian.*

Proof. Clear by the finite dimensionality of A. $\square$

Now we consider the case of rings that satisfy a descending chain condition.

Definition 1.3.3. *An A-module M is said to be* ***Artinian*** *or to satisfy the descending chain condition (DCC) of submodules of M if there exists a k such that*

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_k = M_{k+1} = \cdots .$$

Definition 1.3.4. *A right* ***Artinian ring*** *is a ring A whose right regular module A_A is Artinian.*

A useful criterion is the following:

Proposition 1.9. *Every finitely generated right A-module M over a right Artinian ring A is both Artinian and Noetherian.*

Proof. For a proof see Curtis and Reiner [CR90, page 41] □

As noted in the introduction, one thing that we are interested in is the possible structure of modules given specific simple modules as the building blocks. The following definition begins to shed light on why we view the simple modules as building blocks.

Definition 1.3.5. *A right A-module M has a **composition series** if there exists a descending chain of submodules of M:*

$$M = M_1 \supset M_2 \supset \cdots \supset M_n = 0,$$

*such that the factor modules $\{M_i/M_{i+1} : 1 \leq i < n-1\}$ are simple. The **factors** of the composition series are the M_i/M_{i+1}, and the number of factors is called the **length** of the composition series.*

Proposition 1.10. *A necessary and sufficient condition for a left A-module to have a composition series is that it is both right Noetherian and right Artinian.*

Proof. See Curtis and Reiner [CR90] Section 11. □

Corollary 1.11. *Let G be any finite group. Any finitely generated $\Bbbk G$-module M has a composition series.*

Proof. As $\Bbbk G$ is finite dimensional, it is both Noetherian and Artinian. So by Proposition 1.10, M has a composition series. □

We thus know that the objects we are interested in studying have a composition series. We now would like to have some sort of uniqueness for the composition series. This is given in the following theorem known as the Jordan-Hölder theorem.

Theorem 1.12. *(Jordan-Hölder) Let*

$$0 = M_0 < M_1 < \cdots < M_r = M$$

and

$$0 = N_0 < N_1 < \cdots < N_s = M$$

be two composition series of an A-module M. Then $r = s$ and there exists a permutation ρ such that the composition factors M_i/M_{i-1} and $N_{\rho(i)}/N_{\rho(i)-1}$ are isomorphic as A-modules.

Proof. For a proof see Curtis and Reiner [CR90, page 79]. □

As well as viewing a module in terms of its composition series, in many cases we are interested in decomposing a module into indecomposable modules. We would thus like to know what type of uniqueness we have for decomposition. The following theorem answers this question.

Theorem 1.13. *(Krull-Schmidt) If $M \neq \{0\}$ is an A-module and*

$$M = M_1 \oplus ... \oplus M_r = N_1 \oplus ... \oplus N_s$$

with indecomposable submodules M_i, N_j such that each $\operatorname{End}_A M_i$ and $\operatorname{End}_A N_j$ is local (i.e. has unique maximal two-sided ideal) for $i = 1, ..., r$ and $j = 1, ..., s$ then $r = s$ and there is a permutation $\sigma \in S_r$ with $M_i \cong_A N_{\sigma(i)}$. If $M \neq 0$ is an A-module which is Artinian and Noetherian then M is a finite direct sum of indecomposable A-modules which are uniquely determined up to isomorphism and ordering.

Proof. See Curtis and Reiner [CR90, page 128]. □

Definition 1.3.6. *A right ideal N in a ring A is **nilpotent** if there is a positive integer k such that $N^k = 0$, or equivalently, if $x_1 x_2 \cdots x_k = 0$ for all products of $x_i \in N$. An element $x \in A$ is **nilpotent** if $x^k = 0$ for some k, and a right ideal N is a **nil ideal** if each of its elements is nilpotent.*

In our constructions, we will need to compute the inverse of the sum of a unit and a nilpotent element in a $\Bbbk$-algebra A. The following lemma, which shows the existence gives the algorithm for finding the inverse in the proof.

Lemma 1.14. *Suppose that in a $\Bbbk$-algebra A, we have that $z = r + n$ where r is a unit and n is nilpotent, i.e. there exists an integer s such that $n^s = 0$. Then z is invertible.*

Proof. We wish to find x so that $xz = zx = 1$. We therefore would like to construct $(r+n)^{-1}$. Let

$$x = r^{-1} \cdot \left(1 - \left(\frac{n}{r}\right) + \left(\frac{n}{r}\right)^2 - \cdots \pm \left(\frac{n}{r}\right)^{s-1} + 0\right)$$

If we multiply $(r+n)$ by x we have

$$\begin{aligned} z \cdot x &= (r+n)\, r^{-1} \cdot \left(1 - \left(\frac{n}{r}\right) + \left(\frac{n}{r}\right)^2 - \cdots \pm \left(\frac{n}{r}\right)^{s-1}\right) \\ &= \left(1 + nr^{-1}\right)\left(1 - \left(\frac{n}{r}\right) + \left(\frac{n}{r}\right)^2 - \cdots \pm \left(\frac{n}{r}\right)^{s-1}\right) \\ &= 1 - nr^{-1} + \cdots \pm n^{s-1}r^{-(s-1)} + nr^{-1} + \cdots \mp n^{s-1}r^{-(s-1)} + 0 \\ &= 1. \end{aligned}$$

Similarly, a short computation shows that $xz = 1$. $\square$

The following proposition gives a useful list of properties for Artinian rings.

Proposition 1.15. *Assume that A is a right Artinian ring. Then we have the following:*

1. *The radical of A, $\operatorname{Jac} A$ is nilpotent.*

2. *$A/\operatorname{Jac} A$ is a semisimple ring.*

3. *An A-module M is semisimple if and only if $M \cdot \operatorname{Jac} A = 0$.*

4. There are only a finite number of nonisomorphic simple A-modules.

5. A is right Noetherian.

Proof. For a proof see Auslander [ARS95, pages 9-10]. □

Proposition 1.16. *Let A be a right Artinian ring and I an ideal in A such that I is nilpotent and A/I is semisimple. Then we have $I = \operatorname{Jac} A$.*

Proof. For a proof see Auslander [ARS95, page 10] □

We now have a proposition that lets us relate the radical of a module A to the module times things from the Jacobson radical of the ring A. We will use this fact to help us compute the radical of a module knowing the radical of the respective ring.

Proposition 1.17. *Let M be a finitely generated module over a right Artinian ring A. Then we have $\operatorname{Rad} M = M \cdot \operatorname{Jac} A$.*

Proof. For a proof see Benson [Ben98a, page 4]. □

The last result in this section is often used in representation theory. It will be used to help us prove the existence of projective covers in the following section.

Lemma 1.18. *(Fitting) Suppose that the A-module M has a composition series and $\varphi \in \operatorname{End}_A(M)$. Then for large enough n,*

$$M = \operatorname{Im}(\varphi^n) \oplus \operatorname{Ker}(\varphi^n).$$

Proof. For a proof see Benson [Ben98a, page 8] □

As the $\Bbbk$-algebras that we are mainly interested in are finite dimensional, and all finite dimensional algebras are Artinian, we may use all of the above results for our work. From here on out, we **assume** that all of our $\Bbbk$-algebras A are **finite dimensional** unless otherwise noted.

1.4 Projective Covers

As we will see, one of the important constructions in studying cohomological properties of a module is a projective resolution. We not only want to be able to construct projective resolutions, but we will want to do this in some sort of way that is as small as possible. Thus we first need to define the notion of a projective cover of a module. We will define a projective cover in terms of a special type of epimorphism.

Definition 1.4.1. *Let M and N be A-modules. An epimorphism $\varepsilon : M \twoheadrightarrow N$ is called* ***essential*** *if for each sequence of A-homomorphisms $X \xrightarrow{\tau} M \xrightarrow{\varepsilon} N$ such that $\varepsilon\tau$ is surjective, then τ is also surjective.*

In other words, $\varepsilon : M \to N$ is essential if no proper submodule of M is mapped onto N by ε.

Definition 1.4.2. *A* ***projective cover*** *of an A-module M is a projective module $P(M)$ together with an essential homomorphism $\varepsilon : P(M) \twoheadrightarrow M$.*

This definition is saying that if $P(M) \overset{\varepsilon}{\twoheadrightarrow} M$ is a projective cover of M, then no proper submodule of $P(M)$ is mapped onto M.

Some modules need not have projective covers. For example, the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ has none; for let $f : P \to \mathbb{Z}/2\mathbb{Z}$ be a projective cover with P $\mathbb{Z}$-projective. Then P is $\mathbb{Z}$-free, and $3P$ is a proper submodule of P for which $f(3P) = \mathbb{Z}/2\mathbb{Z}$.

In our case, however, the algebra $\Bbbk G$ is finite dimensional and this is more than enough to ensure the existence of projective covers.

Theorem 1.19. *Let M be a finitely generated A-module. Then M has a projective cover.*

Proof. For a proof see Curtis and Reiner [CR90, pages 132-133]. $\square$

The next thing that we ask is, given a projective cover, is it unique? The answer is in the following proposition.

Proposition 1.20. *Projective covers are unique up to isomorphism, assuming there are any. In other words, given two projective covers* $P \xrightarrow{\varepsilon} M$ *and* $P' \xrightarrow{\varepsilon'} M$*, there exists an isomorphism* $\theta : P \to P'$ *such that* $\varepsilon = \varepsilon'\theta$*.*

Proof. For a proof see [CR90, page 131]. □

Proposition 1.21. *Let* $f : M \twoheadrightarrow N$ *be an epimorphism of finitely generated A-modules. The following are equivalent:*

1. f *is essential.*

2. $\operatorname{Ker} f \subseteq \operatorname{Rad} M$*.*

Proof. This is a consequence of Proposition 1.17. □

As immediate consequences of Proposition 1.21 we have the following. They are important results for us in determining all of the possible PIMs of a module.

Corollary 1.22. *Let* $S_1, ..., S_n$ *be a complete list of nonisomorphic simple A-modules. Then their projective covers* $P_1, ..., P_n$ *are a complete list of nonisomorphic indecomposable projective A-modules (PIMs). Moreover, each* P_i *is isomorphic to a summand of the right regular module* A_A*.*

Corollary 1.23. *Let* P *be a finitely generated projective A-module. Then the natural epimorphism* $P \twoheadrightarrow P/\operatorname{Rad} P$ *gives a projective cover of the A-module* $P/\operatorname{Rad} P$*.*

Proof. The surjection $P \twoheadrightarrow P/\operatorname{Rad} P$ is essential by Proposition 1.21. □

Part of our algorithm will also rely on the fact that there is a 1-1 correspondence between the isomorphism classes of projective indecomposable $\Bbbk G$ modules and the isomorphism classes of simple $\Bbbk G$ modules. Moreover, given a projective indecomposable module P, we shall see that the correspondence is given by $S \cong P/\operatorname{Rad}(P)$. But before we show this, we would like to discuss how we can get representatives for the projective indecomposable modules. This comes from finding a special type of projection operator known as an idempotent. We will discuss these in the next section and then return to giving the proof of the relation between simple modules and their projective covers.

1.5 Idempotents

Suppose $A = \Bbbk G$. As $\Bbbk G$ is finite dimensional, we have a finite number of simple modules (up to isomorphism) from proposition 1.7. We will show that there is a 1-1 correspondence between the PIMs and the simple modules. We would like to explicitly write down this correspondence. We shall see that each projective indecomposable A-module M (up to isomorphism) can be represented as the right module eA where e is a primitive idempotent. We now present the needed definitions and theorems.

Definition 1.5.1. *Suppose A is a $\Bbbk$-algebra. An element $e \in A$ is called an **idempotent** if $e^2 = e$.*

***Example* 1.5.1.** *If we consider all 2×2 matrices over $\mathbb{C}$ we have that*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \textit{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are all idempotents.

***Example* 1.5.2.** *Consider the group algebra of the symmetric group on 3 letters, S_3 over the field $\mathbb{F}_2$. Then $e = 1 \cdot \mathrm{id} + 1 \cdot (123) + 1 \cdot (132)$ is an idempotent as $e^2 = (1 + 1 \cdot (123) + 1 \cdot (132))^2 = 3 \cdot \mathrm{id} + 3 \cdot (123) + 3 \cdot (132) = 1 \cdot \mathrm{id} + 1 \cdot (123) + 1 \cdot (132) = e$.*

Definition 1.5.2. *Two idempotents e and e' in a ring A are said to be **orthogonal** if $ee' = e'e = 0$.*

***Example* 1.5.3.** *In example 1.5.1 we see that*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are clearly orthogonal.

Definition 1.5.3. *We call an idempotent $e \in A$ **primitive** if it cannot be expressed as the sum of two nonzero orthogonal idempotents.*

***Example* 1.5.4.** *$\mathcal{M}$ is a primitive idempotent and I is not where,*

$$\mathcal{M} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition 1.5.4. *A **central idempotent** in A is an idempotent in the center of A.*

Definition 1.5.5. *A **primitive central idempotent** is a central idempotent not expressible as the sum of two orthogonal central idempotents.*

There is a one-one correspondence between expressions $1 = e_1 + \cdots + e_s$ with e_i orthogonal central idempotents and direct sum decompositions

$$A = B_1 \oplus \cdots \oplus B_s \tag{1.4}$$

of A as two-sided ideals, given by $B_i = e_i A$.

Now suppose that A satisfies the D.C.C. Then we can write $A = B_1 \oplus \cdots \oplus B_s$ with the B_i as indecomposable two-sided ideals.

Lemma 1.24. *The decomposition (1.4) above of A into two-sided ideals is unique; i.e. if for some other decomposition $A = B_1^{'} \oplus \cdots B_t^{'}$ then $s = t$ and for some permutation ρ of $\{1, ..., s\}$ we have $B_i = B_{\rho(i)}^{'}$.*

Proof. Write $1 = e_1 + \cdots + e_s = e_1^{'} + \cdots + e_t^{'}$. Then $e_i e_j^{'}$ is also a central idempotent (or zero) for each i, j. Thus $e_i = e_i e_1^{'} + \cdots + e_i e_t^{'}$, so that for a unique j, $e_i = e_i e_j^{'} = e_j^{'}$. $\square$

Definition 1.5.6. *The indecomposable two-sided ideals in this decomposition are called the **blocks** of A.*

Definition 1.5.7. *Suppose M is an indecomposable A-module. Then $M = e_1M \oplus \cdots \oplus e_sM$ shows that for some i, $e_iM = M$ and $e_jM = 0$ for $i \neq j$. We then say that M **belongs to the block** B_i.*

The fact that we have a finite dimensional algebra, allows us (up to isomorphism) to find all of the PIMs. We do this by decomposing the regular representation as in the following theorem.

Theorem 1.25. *Let A be a finite dimensional $\Bbbk$-algebra. Let $S_1, ..., S_r$ be the simple A-modules up to isomorphism and $\Delta_i = \operatorname{End}_A S_i$. Then there are r projective indecomposable modules $P_1, ..., P_r$ (up to isomorphism) with $P_i / \operatorname{Rad} P_i \cong_A S_i$ and*

$$A_A = \bigoplus_{i=1}^{r} (P_{i,1} \oplus ... \oplus P_{i,f_i}) \quad \text{with } P_{i,j} \cong_A P_i$$

where $f_i = \dim_{\Delta_i} S_i$. If $\Bbbk$ is a splitting field for A then the f_i are just the degrees of the irreducible representations of A.

Proof. For a proof see [ARS95, page 14]. □

The simple A-modules $S_i = P_i / \operatorname{Rad} P_i$ and PIMs P_i are classified into blocks. If a module is in a certain block, then so are all its composition factors. Thus if PIMs P_i and P_j (resp. simple modules S_i and S_j) are in different blocks, then there are no possible homomorphisms between P_i and P_j.

Theorem 1.26. *Let P be an indecomposable $\Bbbk G$-module. Then $P = e\Bbbk G$ where e is some primitive idempotent in $\Bbbk G$. The module P has a simple head (top) and a simple socle. Moreover, $P / \operatorname{Rad} P \cong \operatorname{Soc}(P)$.*

Proof. For a proof see Benson [Ben98a, page 12] □

The following theorem gives a way of viewing the homomorphisms from a PIM $e_iA = P_i$ to an A-module M by just looking at the image of the idempotent e_i. This is important in the construction of our maps in our implementations in GAP. This basically means that we can just consider maps by keeping the generators. A generator is just the image of an idempotent for a PIM.

Theorem 1.27. *If $e \in A$ is an idempotent and M is an A-module, then*

1. $\mathrm{Hom}_A(eA, M) \cong Me$ *as* $\Bbbk$*-vector spaces, and*
2. $\mathrm{End}_A(eA) \cong eAe$ *as* $\Bbbk$*-algebras.*

Proof. 1. A natural isomorphism from $\mathrm{Hom}_A(eA, M)$ to Me is given by

$$\varphi \mapsto \varphi(e) = \varphi\left(e^2\right) = \varphi(e)\, e \in Me$$

for $\varphi \in \mathrm{Hom}_A(eA, M)$. The inverse is given as $Me \to \mathrm{Hom}_A(eA, M)$ by

$$ev \longmapsto \varphi_{me} \quad \text{with} \quad \varphi_{me}(ea) = mea \quad \text{for } a \in A,\ m \in M$$

2. Consider the same map defined in (1.) with $M = eA$. We have a ring isomorphism $\mathrm{End}_A(eA) \to eAe$, since

$$\varphi \cdot \psi(e) = \varphi(\psi(e)\, e) = \psi(e)\, \varphi(e) \quad \text{for } \varphi,\ \psi \in \mathrm{End}_A\, eA.$$

□

From the previous theorem 1.27 we can deduce that $\mathrm{Hom}_A(eA, M) \cong Me$ as eAe-modules. We will rely on part (1.) of theorem 1.27 in our algorithms. All of the PIMs (projective indecomposable A-modules) are given to us as eA. These are the projective modules that we will have in our resolutions of simple modules as the PIMs are the projective covers of the simples. We thus will be able to give all of the maps between PIMs just by determining where the idempotents are sent.

There is a close connection between the decomposition of A into a sum of indecomposable A-modules and the decomposition of 1 into a sum of primitive orthogonal idempotents. The next result gives the relation between the simple A-modules and the projective indecomposable A-modules in terms of the primitive idempotents.

We use the following proposition in our construction of a minimal projective resolution using linear algebra.

Proposition 1.28. *Let M be a finitely generated A-module.*

1. *Let $f : P \twoheadrightarrow M$ be an epimorphism with P projective. Then f gives a projective cover of M if and only if* $\operatorname{Ker} f \subseteq P \cdot \operatorname{Jac} A = \operatorname{Rad} P$*.*
2. *For each A-module M, the modules M and $M/\operatorname{Rad} M$ have the same projective cover as A-modules.*
3. *Projective covers are additive, that is, if $f_i : P_i \to M_i$, $1 \leq i \leq k$, are projective covers, then so is*

$$\dot{+}_{i=1}^{k} f_i : \dot{+}_{i=1}^{k} P_i \to \dot{+}_{i=1}^{k} M_i.$$

Proof. For a proof see Curtis and Reiner [CR90, page 133] □

We have a direct sum decomposition of the regular representation of A as

$$A_A = \bigoplus_{i=1}^{r} n_i P_i$$

with $P_i/\operatorname{Rad} P_i \cong S_i$ by Theorem 1.25. By the Krull-Schmidt theorem 1.13 we know that every PIM is isomorphic to one of the $P_i = e_i A$ for a primitive idempotent. The set $\{e_i\}$ of primitive idempotents that we choose are necessarily orthogonal. We now know the PIMs and their corresponding simple modules which are their heads. Lastly, we describe the homomorphisms.

Lemma 1.29.

$$\mathrm{Hom}_A\left(P_i, S_j\right) \cong \begin{cases} \Delta_i, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

where Δ_i is a division ring such that $\Delta_i \cong \mathrm{End}_A\left(S_i\right)$.

Proof. P_i has a unique top composition factor, and this is isomorphic to S_i and therefore by Schur's Lemma 1.1 the result follows. □

Lemma 1.30. *Given PIMs P_i and P_j (given as projective covers for simples S_i and S_j) we have that $\dim_{\Delta_i} \mathrm{Hom}_A\left(P_i, P_j\right)$ is the multiplicity of the simple module S_i as a composition factor of P_j.*

Proof. For a proof see Benson [Ben98a, page 14]. □

The last lemma motivates the following definition.

Definition 1.5.8. *Let $S_1, ..., S_r$ be a complete set of simple A-modules and $P_i = P\left(S_i\right)$ be the projective cover of S_i for $i = 1, ..., r$. Let $c_{i,j}$ be the number of composition factors in a fixed composition series of P_i which are isomorphic to S_j. Then the $r \times r$ matrix $\left[c_{i,j}\right]$ is called the **Cartan matrix** of A.*

To get a feel for idempotents, projective indecomposable modules, and radicals we give an example.

***Example* 1.5.5.** *Let $G = S_3$ and $\Bbbk$ a finite field of characteristic 3. There are exactly two simple $\Bbbk S_3$-modules M_1 and M_2 both of dimension 1. M_1 the trivial representation and M_2 the module afforded by the sign representation, $g \mapsto \mathrm{sign}\left(g\right)$. We know that there must be primitive idempotents e_1 and $e_2 \in \Bbbk S_3$ such that*

$$\Bbbk S_3 = e_1 \Bbbk S_3 \oplus e_2 \Bbbk S_3.$$

It is not difficult to find such idempotents. We wish to investigate the precise structure of the projective indecomposable modules $e_1 \Bbbk S_3$ and $e_2 \Bbbk S_3$. Let $e_1 = \frac{1}{2}\left(1 + (12)\right)$ and find $P_1 = e_1 \Bbbk G$ is:

$$\begin{aligned} P_1 &= \operatorname{Span}_{\Bbbk}(e_1,\, e_1(123),\, e_1(132)) \\ \operatorname{Rad}(P_1) &= \operatorname{Span}_{\Bbbk}(e_1 - e_1(132),\, \boldsymbol{s}_G) \\ \operatorname{Soc}(P_1) &= \operatorname{Span}_{\Bbbk}(\boldsymbol{s}_G) \end{aligned}$$

with $\boldsymbol{s}_G = \sum_{g\in G} g = 2\left(e_1 + e_1(123) + e_1(132)\right)$. *We have that* $\boldsymbol{s}_G$ *is the module identified with the trivial representation and we have that* $P_1/Rad(P_1) \cong \operatorname{Soc}(P_1)$. *Similarly as* $e_1(1-e_1) = e_1 - e_1^2 = e_1 - e_1 = 0$ *we take* $e_2 = 1 - e_1 = \frac{1}{2}(1-(12))$. *We have that* $P_2 = e_2\Bbbk G$ *is:*

$$\begin{aligned} P_2 &= \operatorname{Span}_{\Bbbk}(e_2,\, e_2(123),\, e_2(132)) \\ \operatorname{Rad}(P_2) &= \operatorname{Span}_{\Bbbk}(1 + (123) + (132),\, \boldsymbol{a}_G) \\ \operatorname{Soc}(P_2) &= \operatorname{Span}_{\Bbbk}(\boldsymbol{a}_G) \end{aligned}$$

with $\boldsymbol{a}_G = \sum_{g\in G} sgn(g)\, g$. *It is easy to see that the composition factors of* P_1 *and* P_2 *are* $M_1,\ M_2,\ M_1$ *and* $M_2,\ M_1,\ M_2$ *respectively.*

If $\operatorname{char}\Bbbk = 2$ *then* $\Bbbk G$ *has a simple module* $M_1 = \Bbbk_G$, M_2 *with* $\dim_{\Bbbk} M_2 = 2$. *Thus*

$$\Bbbk G \cong P(M_1) \oplus P(M_2) \oplus P(M_2) \quad \textit{with} \quad \dim_{\Bbbk} P(M_i) = 2 \ (i = 1, 2).$$

The Cartan matrices of $\Bbbk S_3$ *for* $\Bbbk = \mathbb{F}_2$ *and* $\mathbb{F}_3$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} (\operatorname{char}\Bbbk = 2) \qquad \textit{and} \qquad C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} (\operatorname{char}\Bbbk = 3).$$

1.6 Basic Algebras

As noted in the introduction, we are interested in studying some rather large algebras via computational methods on a computer. As the dimension of the finite dimensional algebra $\Bbbk G$ is the size of the group G, this can be a rather large object to deal with on a computer. For example the Higman Sims group is one of the small to medium sized sporadic simple groups with $|G| = 44,352,000$. Its group algebra is too large to do any meaningful computations with it. In many instances it is technically and computationally easier to deal with modules over finite dimensional $\Bbbk$-algebras A which have the property that if $A = \bigoplus_{i=1}^{n} P_i$ with the P_i projective indecomposable modules (PIMs), then $P_i \ncong_A P_j$ for $i \neq j$. Such algebras A are called **basic algebras**. The process of computing the basic algebra of group algebras has been implemented in a GAP package by T. Hoffman [Hof04]. His work gives us a large data base for basic algebras of group algebras and the ability to compute more. Our implementation begins by having a basic algebra, however, we include the results about basic algebras for completeness.

***Example* 1.6.1.** *Consider a p-group G and $\Bbbk = \mathbb{F}_p$. Then as the only simple $\Bbbk G$-module is the trivial module, $\Bbbk G$ is a basic algebra.*

Before we go further into detail about basic algebras and their constructions, we first need to define some basic notions from category theory.

1.6.1 Category Theory

We now introduce some basic notions from category theory that will be needed in discussing the equivalence of the category of finitely generated $\Bbbk G$-modules ($\mathrm{mod}_{\Bbbk G}$) and finitely generated B-modules (mod_B) where B is a basic algebra. A good reference for this material is Hilton and Staumbach [HS97].

Definition 1.6.1. *A **category** $\mathfrak{C}$ has three pieces of data:*

1. *A class of objects* $\mathsf{Obj}\,(\mathfrak{C})$,

2. *For each pair* $M, N \in \mathsf{Obj}\,(\mathfrak{C})$, *a set* $\mathfrak{C}\,(M,N)$ *of morphisms from* M *to* N,

3. *The set* $\mathfrak{C}\,(M_1, M_2) \times \mathfrak{C}\,(M_2, M_3)$ *consists of pairs* (f, g) *where* $f : M_1 \to M_2$ *and* $g : M_2 \to M_3$ *and we write the composition of* f *and* g *as* $g \circ f$. *The composite function* $g \circ f$ *is the function* h *from* M_1 *to* M_3 *given by*

$$h(a) = g(f(a)), \qquad a \in M_1.$$

For each triple $M_1, M_2, M_3 \in \mathrm{Obj}\,(\mathfrak{C})$,*there is a law of composition* $g \circ f$

$$\mathfrak{C}\,(M_1, M_2) \times \mathfrak{C}\,(M_2, M_3) \to \mathfrak{C}\,(M_1, M_3)$$

which satisfy the following axioms:

A1. *The sets* $\mathfrak{C}\,(M_1, N_1)$ *and* $\mathfrak{C}\,(M_2, N_2)$ *are disjoint unless* $M_1 = M_2$ *and* $N_1 = N_2$;

A2. *The morphisms* $f \in \mathfrak{C}\,(M_1, M_2)$, $g \in \mathfrak{C}\,(M_2, M_3)$ *and* $h \in \mathfrak{C}\,(M_3, M_4)$ *satisfy the associative law of composition, i.e.,*

$$h\,(gf) = (hg)\,f;$$

A3. *There is a morphism* $1_M : M \to M$ *such that, for any* $f : M \to N_1$, $g : N_2 \to M$,

$$f 1_M = f \qquad \textit{and} \qquad 1_M g = g,$$

for all $M, M_1, M_2, M_3, M_4, N_1, N_2 \in \mathrm{Obj}\,(\mathfrak{C})$.

In our work, we are interested in finitely generated A-modules.

***Example* 1.6.2.** *Let* A *be an finite dimensional algebra (a group algebra in our case). We denote the category of finitely generated* A*-modules by* mod_A. *The objects we take are finitely generated* A*-modules,* M, N, *and the morphisms are the* A*-homomorphisms,* $\mathrm{Hom}_A\,(M, N)$.

We are also interested in the relationship between categories.

Definition 1.6.2. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. A* ***functor*** $F : \mathfrak{C} \to \mathfrak{D}$ *is a rule which assigns to each object* $M \in \mathrm{Obj}\,(\mathfrak{C})$ *an object* $F(M) \in \mathrm{Obj}\,(\mathfrak{D})$ *and to each morphism* $f \in \mathfrak{C}\,(M, N)$ *a morphism* $F(f) \in \mathfrak{D}\,(F(M), F(N))$, *such that*

$$F\,(fg) = F(f)F(g),$$

for $M, N, O \in \mathrm{Obj}\,(\mathfrak{C})$, $f \in \mathfrak{C}\,(M, N)$ *and* $g \in \mathfrak{C}\,(O, M)$, *and*

$$F\,(1_M) = 1_{F(M)}.$$

Definition 1.6.3. *Let F and G be functors from the category $\mathfrak{C}$ to the category $\mathfrak{D}$. Then a* ***natural transformation*** *t from F to G is a rule assigning to each object $M \in \mathfrak{C}$ a morphism $t_M : F(M) \to G(M)$ in $\mathfrak{D}$ such that for any morphism $f \in \mathfrak{C}\,(M, N)$, the diagram*

$$\begin{tikzcd}
F(M) \arrow[r, "t_M"] \arrow[d, "F(f)"'] & G(M) \arrow[d, "G(f)"] \\
F(N) \arrow[r, "t_N"] & G(N)
\end{tikzcd}$$

commutes. If t_M is an isomorphism for every $M \in \mathfrak{C}$, then t is called a ***natural equivalence****, and the functors F and G are said to be naturally equivalent.*

Definition 1.6.4. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two categories. We call $\mathfrak{C}$ and $\mathfrak{D}$* ***equivalent*** *if there exist functors*

$$F : \mathfrak{C} \to \mathfrak{D}$$

and

$$G : \mathfrak{D} \to \mathfrak{C}$$

such that $F \circ G$ and $G \circ F$ are naturally equivalent to the identity functors of $\mathfrak{D}$ and $\mathfrak{C}$, respectively.

***Example* 1.6.3.** *Let $\mathfrak{B}_{\Bbbk}$ denote the category of finite dimensional vector spaces over the field $\Bbbk$ with linear transformations as the morphisms. Let V be a vector space over a field $\Bbbk$, let V^* be the dual vector space and V^{**} be the double dual. There is a linear map $\iota_V : V \to V^{**}$ given by $v \mapsto \tilde{v}$ where $\tilde{v}(\varphi) = \varphi(v)$, $v \in V$, $\varphi \in V^*$, and $\tilde{v} \in V^{**}$. Then ι is a natural transformation from the identity functor $I : \mathfrak{B}_{\Bbbk} \to \mathfrak{B}_{\Bbbk}$ to the double dual functor $** : \mathfrak{B}_{\Bbbk} \to \mathfrak{B}_{\Bbbk}$.*

1.6.2 Morita Theory

Definition 1.6.5. *We call the finite dimensional algebras A and B* ***Morita equivalent*** *if the categories mod_A and mod_B are equivalent.*

When we compute the Morita equivalent basic algebra B of a group algebra $\Bbbk G$ we lose information about the group, however, we keep many important properties as Morita equivalence is a strong equivalence. The following lemma gives many of the properties that are preserved.

Lemma 1.31. *Let A and B be Morita equivalent algebras with $F : \text{mod}_A \to \text{mod}_B$ and $G : \text{mod}_B \to \text{mod}_A$ the functors for this equivalence. Then the following hold for M, M', M'' in mod_A.*

1. *The sequence*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

   is (split) exact if and only if the sequence

$$0 \longrightarrow F(M') \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(M'') \longrightarrow 0$$

   is (split) exact.

2. *M is projective if and only if $F(M)$ is projective.*

3. *$f : M \to M'$ is a projective cover if and only if $F(f) : F(M) \to F(M')$ is a projective cover.*

4. *M is simple (semisimple) if and only if $F(M)$ is simple (semisimple).*

5. *M is indecomposable if and only if $F(M)$ is indecomposable.*

Furthermore, the lattice of submodules of M is isomorphic to the lattice of submodules of $F(M)$. This implies that $F(\mathrm{Rad}(M)) = \mathrm{Rad}(F(M))$. We also know that for Morita equivalent algebras A and B, the number of isomorphism classes of simple modules is the same.

Proof. Proofs of the statements in this lemma can be found in [AF92, pages 254–258]. □

In general, we are not in the situation of the above example, i.e., we start with an algebra that is not basic and want to construct a basic algebra that is Morita equivalent to our original algebra. The following gives a method of constructing the basic algebra.

Theorem 1.32. *Let A be a finite dimensional algebra. Let $S_1, ..., S_t$ be the simple A-modules (up to isomorphism) and for each $i = 1, ..., t$ let P_i be the projective cover of S_i. Let*

$$P = \dot{+}_{i=1}^{t} P_i$$

and

$$B = \mathrm{End}_A(P, P) = \dot{+}_{i,j=1}^{t} \mathrm{Hom}_A(P_i, P_j).$$

Then B is a basic algebra that is Morita equivalent to A.

Proof. For a proof see [ARS95, pages 35-36] □

***Example* 1.6.4.** *Consider the $\Bbbk$-algebra of all 2×2 matrices over $\Bbbk$ and denote it by $\mathcal{M}$. Then we can take $\mathcal{M}$ as a right $\mathcal{M}$ module with an action of right multiplication. Up to isomorphism there is one PIM P with matrices of the form:*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

Then $\mathrm{End}(P) = \Bbbk$ *is Morita equivalent to $\mathcal{M}$.*

We will see later that the basic algebra B of a group algebra $\Bbbk G$ will be the starting point for us in our algorithm of computing the cohomology ring and Ext-algebra of group algebras. What we will do is simply compute the Ext-algebra for B and we will be able to derive from lemma 1.31 that there is an isomorphism to the Ext-algebra of $\Bbbk G$ (see Theorem 2.11). To end this section, we include a result we use in Theorem 2.11.

Proposition 1.33. *Let $F : \mathrm{mod}_A \to \mathrm{mod}_B$ and be $G : \mathrm{mod}_B \to \mathrm{mod}_A$ be the functors for the equivalence of Morita equivalent finite dimensional algebras A and B. Then for each M, N in* mod_A *the restriction of F to* $\mathrm{Hom}_A(M, N)$ *is an abelian group isomorphism*

$$F : \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_B(F(M), F(N))$$

such that $F(f)$ is an epimorphism (monomorphism) in mod_B *if and only if f is an epimorphism (monomorphism) in* mod_A. *Moreover, if $M \neq 0$, then this restriction*

$$F : \mathrm{End}_A(M) \longrightarrow \mathrm{End}_B(F(M))$$

is a ring isomorphism.

Proof. See Anderson and Fuller [AF92, page 252]. □

1.7 Quivers and Path Algebras

Our alternative approach to using linear algebra to compute projective resolutions will use Gröbner basis theory. We wish to develop a theory for Gröbner bases over path algebras, which are generally noncommutative. The reason we use this approach is that from Gabriel's theorem 3.1, a basic algebra can be given as the quotient of a path algebra $\Bbbk\Gamma$ by a relations ideal I contained in the ideal generated by paths of length two. We first present the basic definitions and theorems from path algebras.

Definition 1.7.1. *A* ***quiver*** Γ *is a directed graph. Loops and multiple edges are allowed. The edges are called* ***arrows.*** *Each arrow* a *is directed so it has an* ***origin vertex*** $o(a)$ *and a* ***terminus vertex*** $\tau(a)$. *A* ***finite quiver*** *is a quiver with finitely many arrows and vertices. A* ***path*** *in* Γ *of length* l *is a sequence of arrows* $a_1, ..., a_l$ *such that* $\tau(a_i) = o(a_{i+1})$ *for* $1 \leq i \leq l-1$. *The path is denoted* $a_1 \cdots a_l$. *For each vertex* v *there is a vertex path of the same name with length* 0 *such that* $v^2 = v$.

We shall assume that all of our quivers are finite unless otherwise noted. Next we describe a way of giving a quiver Γ an algebra structure.

Definition 1.7.2. *A* ***path algebra*** $\Bbbk\Gamma$ *over a field* $\Bbbk$ *is the* $\Bbbk$*-algebra with a* $\Bbbk$*-basis consisting of the finite directed paths in* Γ. *Thus, elements of* $\Bbbk\Gamma$ *are the* $\Bbbk$*-linear combinations of paths in* Γ. *We define a multiplication on paths* p *and* q *by concatenation* pq *if* $\tau(p) = o(q)$ *and as* 0 *otherwise. We view the vertices as paths of length* 0 *with multiplication given as follows. If* v *and* w *are vertices and* p *is a path, we let* $v \cdot w$ *be* v *if* $v = w$ *and* 0 *otherwise. We let* $v \cdot p = p$ *if* v *is the origin of* p *and* 0 *otherwise, and we define* $p \cdot w$ *similarly. The multiplication on paths is extended linearly to arbitrary elements of* $\Bbbk\Gamma$.

Example **1.7.1.** *For* $n \geq 1$, *let* Γ *be the quiver with one vertex* v *and* n *arrows* $a_1, ..., a_n$, *all loops at* v. *Then the path algebra* $\Bbbk\Gamma$ *is the free associative algebra* $\Bbbk \langle a_1, ..., a_n \rangle$.

Example **1.7.2.** *The following is an example of a finite dimensional path algebra. Note that to be finite dimensional there can be no loops. Let* $\Bbbk$ *be any field. Let* Γ *be:*

$$\underset{v_3}{\bullet} \xrightarrow{a} \underset{v_1}{\bullet} \xrightarrow{b} \underset{v_2}{\bullet}$$

A basis for the path algebra $\Bbbk\Gamma$ *is*

$$\{v_1, v_2, v_3, a, b, ab\}$$

As mentioned previously, we have a relation between the basic algebra B of a group algebra $\Bbbk G$ and the quotient of a path algebra. Thus the motivation for the use of Gröbner basis theory comes from the following theorem.

Theorem 1.34. *Let* $\Bbbk$ *be a splitting field for* Λ *a finite dimensional basic* $\Bbbk$*-algebra. Then there is a finite directed graph* Γ *and an ideal* I *contained in the ideal generated by the paths of length* 2 *such that* $\Lambda = \Bbbk\Gamma/I$*.*

Proof. For a proof see Benson [Ben98a, page 103]. $\square$

Although this is merely an existence theorem, as a result of the ideas in section 1.6, there is a constructive method for finding such a graph Γ and an ideal I. See Hoffman [Hof04] and Theorem 3.1.

1.7.1 Ideals in Path Algebras

Throughout this section we let $\Bbbk$ be a field, Γ be a finite quiver, I be an ideal in the path algebra $\Bbbk\Gamma$, and J the ideal in $\Bbbk\Gamma$ generated by the arrows of Γ.

Definition 1.7.3. *Let* I *be an ideal in the path algebra* $\Bbbk\Gamma$*. If there exists* $N \geq 2$ *such that* $J^N \subseteq I \subseteq J^2$ *then we call the pair* (Γ, I) *a* ***special quiver with relations****. We shall denote the quotient algebra* $\Bbbk\Gamma/I = \Lambda$*.*

Remark 1.7.1. *The standard definition of a quiver with relations does not require* Γ *to be finite, nor does it demand that some* J^N *be contained in* I*. Hence the word "special."*

The following is an example of a special quiver with relations.

***Example* 1.7.3.** *Let $\Bbbk = \mathbb{F}_2$ and Γ given as*

$$v_1 \underset{b}{\overset{a}{\rightleftarrows}} v_2$$

Let $I = \langle aba, bab \rangle$. Then (Γ, I) is a special quiver with relations.

Definition 1.7.4. *Let $\Bbbk\Gamma$ be a path algebra, and v a vertex of Γ. The **vertex simple module** of $\Bbbk\Gamma$ associated to v is one-dimensional and v acts on it as the identity. The remaining vertices lie in the annihilator of this module, as do the arrows. Denote this module by S_v.*

Note that for S_v a vertex simple $\Bbbk\Gamma$-module, if (Γ, I) is a special quiver with relations then S_v is also a simple $\Lambda = \Bbbk\Gamma/I$-module. We also refer to it as a vertex simple for Λ. This is true as we have that $I \subseteq \operatorname{Ann}(S_v)$, the annihilator of S_v, i.e. all $x \in \Bbbk\Gamma$ such that $S_v \cdot x = 0$.

Lemma 1.35. *Let $\Bbbk$ be a field, let Γ be a quiver, and let (Γ, I) be a special quiver with relations. Then the following hold for $\Lambda = \Bbbk\Gamma/I$:*

1. *The $\Bbbk$-algebra Λ is finite-dimensional. The ideal J/I is the Jacobson radical of Λ, and its nilradical.*

2. *The simple Λ-modules are in one-one correspondence with the vertices of Γ. For a vertex v, the vertex simple S_v has projective cover $e_v\Lambda$. The map $\varepsilon : e_v\Lambda \to S_v$ is given for $a \in e_v\Lambda$ by $a \mapsto a + \operatorname{Rad} e_v\Lambda$.*

Proof. For a proof see Green [Gre97, page 9]. $\square$

Chapter 2
Cohomology and Ext

Recall that according to Maschke's theorem 1.3 if the characteristic p of the field $\Bbbk$ does not divide the order of G, then we know that all $\Bbbk G$-modules are semisimple. When p does divide the order of G, this is no longer true. In this situation, a new class of interesting modules arises which are no longer semisimple. However, any $\Bbbk G$-module still has a composition series. The reconstruction of a $\Bbbk G$-module in the case where p divides the order of the group G from simple composition factors is far more complicated. As we previously mentioned, this is a difficult task that we call the extension problem. The approach we take to studying the extension problem is applying methods from homological algebra.

The definition of an Ext-algebra may be given in terms of equivalence classes of long exact sequences which is a useful theoretical tool (for more see [HS97, pages 84-94,148-155]). However, for computational purposes a more practical way of describing the Ext-algebra is by using minimal projective resolutions. The outline of a specific computational implementation using projective resolutions was first sketched in 1997 by Carlson, Green, and Schneider [CGS97]. The literature covers two generally different ways of carrying out the computation of projective resolutions; one using linear algebra and one using Gröbner basis theory. In this chapter we focus on the linear algebra approach and in chapter 3 we outline the method using Gröbner basis theory. Before we outline the linear algebra approach, we review some basics from homological algebra.

2.1 Homological Algebra

One of our ultimate goals in this dissertation is to make a cohomological computation for the Morita equivalent basic algebra B of our given finite dimensional algebra $\Bbbk G$. To do this we will need to define the notion of a projective resolution, group cohomology, and Ext-algebra.

Definition 2.1.1. *A* ***chain complex*** *C over A is a collection of right A-modules C_n indexed by $\mathbb{Z}$ with homomorphisms $\partial_n : C_n \to C_{n-1}$ such that $\partial_n \circ \partial_{n+1} = 0$.*

Definition 2.1.2. *Let C and D be chain complexes. A* ***chain map*** *$f : C \to D$ consists of A-module homomorphisms $f_n : C_n \to D_n$, with $n \in \mathbb{Z}$, such that the diagrams*

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "\partial'_{n-1}"] & D_{n-1} \end{tikzcd}$$

commute for all n.

Next, we define one of the most important notions of homology. Let C be a chain complex over A. The condition $\partial_n \circ \partial_{n+1} = 0$ implies that $\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n$. To measure how close a chain complex is to being an exact sequence we make the following definition.

Definition 2.1.3. *Given a chain complex C over A we define the (n-th)* ***homology module*** *of C as*

$$H_n(C) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}.$$

Definition 2.1.4. *A* ***cochain complex*** *C over A is a collection of right A-modules C^n indexed by $\mathbb{Z}$ with homomorphisms $\partial^n : C^n \to C^{n+1}$ such that $\partial^n \circ \partial^{n-1} = 0$.*

Cochain maps are defined analogously to chain maps with the arrows being reversed. Similarly we define the cohomology module and the rest of the definitions we make with chains are made for cochains.

Definition 2.1.5. *Let C and D be chain complexes with chain maps $f, g : C \to D$. We call f and g* ***chain homotopic*** *if there exist homomorphisms $h_n : C_n \to D_{n+1}$, with $n \in \mathbb{Z}$, such that $f_n - g_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n$.*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

The chain complexes C and D are called ***chain homotopy equivalent*** *if there are chain maps $f : C \to D$ and $g : D \to C$ such that $f \circ g$ and $g \circ f$ are chain homotopic to the chain maps id_D and id_C respectively.*

We note that chain homotopy is an equivalence relation on chain complexes.

Proposition 2.1. *Let C and D be chain (cochain) complexes. If C and D are chain (cochain) homotopy equivalent then $H_n(C) \cong H_n(D)$ (resp. $H^n(C) \cong H^n(D)$) for all $n \in \mathbb{Z}$.*

Proof. For a proof see [HS97, page 124]. $\square$

2.2 Projective Resolutions

Definition 2.2.1. *Let M be an A-module. A* ***projective resolution*** *$(P_\bullet, \varepsilon)$ of M is an exact sequence of projective modules P_i:*

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

The first important thing to note here is that projective resolutions always exist. This is because every A-module M is the quotient of a free A-module and all free modules are projective.

***Example* 2.2.1.** *Suppose that G is a cyclic p-group of order p^n, $G = \langle x : x^{p^n} = 1 \rangle$. Let $\mathfrak{N}_G = \sum_{g \in G} g$ be the sum of the elements in G. Then we have a periodic projective resolution $(P_\bullet, \varepsilon)$ of the trivial module $\Bbbk$ of the form*

$$\cdots \xrightarrow{\mathfrak{N}_G} P_3 \xrightarrow{x-1} P_2 \xrightarrow{\mathfrak{N}_G} P_1 \xrightarrow{x-1} P_0 \xrightarrow{\varepsilon} \Bbbk \longrightarrow 0$$

where $P_i \cong \Bbbk G$ for every i. That is, the boundary map on P_i for i odd is multiplication by $x - 1$ and for i even it is multiplication by $\mathfrak{N}_G$. The exactness of this resolution can be checked by noting that the elements

$$\left\{ 1, x-1, (x-1)^2, ..., (x-1)^{p^n-2}, \mathfrak{N}_G \right\}$$

form a $\Bbbk$-basis for the free $\Bbbk$-module P_i for every i.

We next compare how two resolutions of an A-module M are related. This is answered by the following proposition.

Proposition 2.2. *Two projective resolutions $(P_\bullet, \varepsilon)$ and $(P'_\bullet, \varepsilon')$ of an A-module M are homotopy equivalent.*

Proof. The proof of this proposition is found in [HS97, page 129]. $\square$

Thus any two resolutions of M are equally good from a theoretical point of view. However free resolutions tend to grow rather quickly in the case of group algebras. Thus when we consider computing resolutions we would like to have some notion of minimality and would like to find a way to compute a minimal resolution. The minimal resolutions will be the ones which have the smallest possible $\Bbbk$-dimension of each projective module in the resolution.

Definition 2.2.2. *Let P be a projective resolution for the A-module M. We call $(P_\bullet, \varepsilon)$ a **minimal projective resolution** for M if for all $n \in \mathbb{Z}^+$ we have*

$$\operatorname{Im} \partial_n \subseteq \operatorname{Rad} P_{n-1}.$$

Throughout the construction of our minimal resolutions, we compute kernels of homomorphisms. These kernels are sometime referred to as Heller modules. We now give their definition.

Definition 2.2.3. *Let $(P_\bullet, \varepsilon)$ be a minimal projective resolution of an A-module M. We define the **Heller Module** Ω^n for $n > 0$ as:*

$$\Omega^n(M) := \operatorname{Ker} \partial_{n-1} = \operatorname{Im} \partial_n$$

where $\operatorname{Ker} \partial_0 := \operatorname{Ker} \varepsilon$, and for $n = 1$ sometimes we write $\Omega^1(M)$ as $\Omega(M)$.

Recall that for finitely generated A-modules M, we have the existence of projective covers from theorem 1.19 on page 37. Thus we can use the existence of projective covers to come up with a straightforward method of constructing a minimal resolution. That is, let $\varepsilon : P_0 \to M$ be a projective cover of M. Then the kernel of ε is $\Omega(M)$ which has no projective submodules. In particular, from Propositions 1.21 and 1.5 we know that the inclusion $i_1 : \Omega(M) \to P_0$ has image in $\operatorname{Rad} P_0$. Now let $\omega_1 : P_1 \to \Omega(M)$ be the projective cover of $\Omega(M)$. The kernel of ∂_1 is $\Omega^2(M)$ and the inclusion $i_2 : \Omega^2(M) \to P_1$ has image in the radical. We continue to build a resolution in this fashion. The boundary map $\partial_n : P_n \to P_{n-1}$ is the composition $i_n \circ \omega_n$.

Algorithm 2.2.1. *Minimal Projective Resolution*

Input: *M, an A-module, n the number of steps in the resolution we wish to compute.*

Output: *A projective resolution of M to n steps.*

1: *Compute $P(M)$ the projective cover (unique up to isomorphism) with an essential homomorphism $\varepsilon : P(M) \twoheadrightarrow M$.*

2: *Compute the kernel of ε, $\Omega^1(M)$. Note this is an A-submodule of $P(M)$*

3: *Construct the map $\Omega^1(M) \to P(M)$ which is just the injection map, denoted ι_1.*

4: *Since $\Omega^1(M)$ is also an A-module, it has a projective cover $P(\Omega^1(M))$ with an essential homomorphism $\omega_1 : P(\Omega^1(M)) \twoheadrightarrow \Omega^1(M)$.*

5: We now define $\partial_1 : P\left(\Omega^1\left(M\right)\right) \to P\left(M\right)$ as the composition $\iota_1 \circ \omega_1$.

6: Repeat procedure until we have reached the desired n.

This procedure results in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P(\Omega^2(M)) \arrow[rr, "\partial_2"] \arrow[dr, "\omega_2"] & & P(\Omega^1(M)) \arrow[rr, "\partial_1"] \arrow[dr, "\omega_1"] & & P(M) \arrow[r, "\varepsilon"] & M \\
& & \Omega^2(M) \arrow[ur, "\iota_2"] & & \Omega^1(M) \arrow[ur, "\iota_1"] & &
\end{tikzcd} \tag{2.1}$$

Proposition 2.3. *The above construction in Algorithm 2.2.1 is a minimal projective resolution.*

Proof. Clearly all of the terms P_n are projective by construction. It is also clear that $\operatorname{Im}\partial_n = \operatorname{Ker}\partial_{n-1}$. For minimality, we note that each $P\left(\Omega^n(M)\right)$ is a projective cover. Thus by definition each map ω_n is essential. By Proposition 1.21 we know that $\operatorname{Ker}\left(\omega_n\right) \subseteq \operatorname{Rad} P\left(\Omega^n(M)\right)$. As each map ι_n is injective, we know that $\operatorname{Ker}\partial_n = \operatorname{Ker}\omega_n$ for each n. Thus

$$\operatorname{Im}\partial_n = \operatorname{Ker}\omega_{n-1} \subseteq \operatorname{Rad} P\left(\Omega^{n-1}(M)\right).$$

□

2.3 The Ext-Algebra and Cohomology Ring

Our ultimate goal is to compute the cohomology ring and the Ext-algebra of a finite dimensional algebra A. Let M and N be A-modules, and suppose

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is a projective resolution for M which we denote $(P_\bullet, \varepsilon)$. We may form a related sequence by taking homomorphisms of each of the terms into N, keeping in mind that this reverses the direction of the homomorphisms in the resolution. We obtain the sequence:

$$0 \to \mathrm{Hom}_A(M,N) \xrightarrow{\varepsilon^*} \mathrm{Hom}_A(P_0,N) \xrightarrow{\partial_1^*} \mathrm{Hom}_A(P_1,N) \xrightarrow{\partial_2^*} \mathrm{Hom}_A(P_2,N) \longrightarrow \cdots$$

and ∂_n^* and ε^* denote the induced maps from $\mathrm{Hom}_A(P_{n-1},N)$ to $\mathrm{Hom}_A(P_n,N)$ induced from ∂_n and ε. The sequence is not necessarily exact, however, it is a cochain complex. The corresponding cohomology groups have a special name.

Definition 2.3.1. *Let M and N be A-modules. Let $(P_\bullet,\varepsilon)$ be a projective resolution of M. Let $\partial_n^* : \mathrm{Hom}_A(P_{n-1},N) \to \mathrm{Hom}_A(P_n,N)$ as above. The group $\mathrm{Ext}_A^n(M,N)$ for $n \geq 0$ is called the n^{th}* ***cohomology group*** *derived from the functor $\mathrm{Hom}_A(-,N)$ and is defined as:*

$$\mathrm{Ext}_A^n(M,N) = \frac{\mathrm{Ker}\,\partial_{n+1}^*}{\mathrm{Im}\,\partial_n^*} = H^n(\mathrm{Hom}_A(P_\bullet,N)),$$

where $\mathrm{Ext}_A^0(M,N) = \mathrm{Ker}\,\partial_1^* \cong \mathrm{Hom}_A(M,N)$.

The first important thing that we note from a computational point of view is the this group is independent of the choice of resolution, and thus we would always like to use a minimal resolution if possible.

Proposition 2.4. *The groups $\mathrm{Ext}_A^n(M,N)$ depend only on M and N, i.e., they are independent of the choice of projective resolution of M.*

Proof. Assume that we have two projective resolutions $(P_\bullet,\varepsilon)$ and $(P'_\bullet,\varepsilon')$ of M:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] \arrow[d, shift left, "f_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, shift left, "f_0"] & M \arrow[r] \arrow[d, shift left, "\mathrm{id}"] & 0 \\
\cdots \arrow[r, "\partial'_2"] & P'_1 \arrow[r, "\partial'_1"] \arrow[u, shift left, "g_1"] & P'_0 \arrow[r, "\varepsilon'"] \arrow[u, shift left, "g_0"] & M \arrow[r] \arrow[u, shift left, "\mathrm{id}"] & 0
\end{tikzcd} \tag{2.2}$$

$(P_\bullet,\varepsilon)$ and $(P'_\bullet,\varepsilon')$ are homotopy equivalent by Proposition 2.2 and so there are chain maps f and g such that $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$ up to homotopy as in (2.2). The commutative diagram in (2.2) implies that the induced diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(M,N) \arrow[r] \arrow[d, shift left, "\mathrm{id}^*"] & \mathrm{Hom}_A\left(P_0,N\right) \arrow[r] \arrow[d, shift left, "g_0^*"] & \mathrm{Hom}_A\left(P_1,N\right) \arrow[r] \arrow[d, shift left, "g_1^*"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_A(M,N) \arrow[r] \arrow[u, shift left, "\mathrm{id}^*"] & \mathrm{Hom}_A\left(P_0',N\right) \arrow[r] \arrow[u, shift left, "f_0^*"] & \mathrm{Hom}_A\left(P_1',N\right) \arrow[r] \arrow[u, shift left, "f_1^*"] & \cdots
\end{tikzcd}$$

is also commutative and $f^* \circ g^* = \mathrm{id}^*$ and $g^* \circ f^* = \mathrm{id}^*$ up to homotopy. Therefore by Proposition 2.1 we have an isomorphism of cohomology. □

We are not only interested in computing minimal projective resolutions to keep growth as small as possible, but also because it greatly simplifies our calculations by not having to worry about coset representatives.

Proposition 2.5. *Let $(P_\bullet, \varepsilon)$ be a projective resolution of a finitely generated A-module M. Then the following statements are equivalent.*

1. *$(P_\bullet, \varepsilon)$ is a minimal projective resolution of M.*

2. *If S is a simple A-module, then for all $n \geq 0$*

$$\mathrm{Hom}_A\left(P_n, S\right) = \mathrm{Ext}_A^n\left(M, S\right).$$

3. *If S is a simple A-module, then for every $n \geq 0$*

$$\partial_n^* : \mathrm{Hom}\left(P_n, S\right) \longrightarrow \mathrm{Hom}\left(P_{n+1}, S\right)$$

is the zero map.

Proof. (1) $\implies$ (3) Assume that $(P_\bullet, \varepsilon)$ is a minimal resolution of M and let S be any simple A-module. Then for any $n \geq 0$,

$$\partial_{n+1}\left(P_{n+1}\right) \subseteq \mathrm{Rad}\, P_n.$$

So if we have a map $\alpha : P_n \to S$, then

$$\alpha\partial_{n+1}\left(P_{n+1}\right) \subseteq \mathrm{Rad}\, S = \{0\},$$

as S is simple. Therefore $\operatorname{Im} \partial^* (\alpha) = 0$ and so (1) implies (3).

(3) $\implies$ (1) Assume that every map $\partial_n^* : \operatorname{Hom}(P_n, S) \to \operatorname{Hom}(P_{n+1}, S)$ is the zero map. Then given a map $\varphi : P_n \to S$ it must be true that $\partial_{n+1}(P_{n+1}) \subseteq \operatorname{Ker} \varphi$. Therefore $\partial_{n+1} \subseteq \operatorname{Rad} P_n$. As this is true for arbitrary n, we have that the resolution $(P_\bullet, \varepsilon)$ must be a minimal resolution.

(3) $\implies$ (2) If statement (3) is true, then for any simple module S,

$$\begin{aligned} \operatorname{Ext}_A^n (M, S) &\cong \operatorname{Ker} \partial_{n+1}^* / \operatorname{Im} \partial_n^* \\ &\cong \operatorname{Hom}_A (P_n, S) / \{0\} \\ &\cong \operatorname{Hom}_A (P_n, S) . \end{aligned}$$

Thus (3) implies (2).

(2) $\implies$ (3) If we assume (2), then

$$\begin{aligned} \operatorname{Hom}_A (P_n, S) &= \operatorname{Hom}_A (P_n, S) / \{0\} \\ &= \operatorname{Hom}_A (P_n, S) / \partial_n^* (\operatorname{Hom}_A (P_{n-1}, S)) \\ &= \operatorname{Ext}_A^n (M, S) . \end{aligned}$$

Therefore, (2) implies (3). $\square$

The next thing we introduce is a multiplication of elements of $\operatorname{Ext}_A^n (M, N)$ and $\operatorname{Ext}_A^m (N, L)$. We want to be able to multiply extensions to give a ring structure, so we want a well-defined bilinear, associative map:

$$\operatorname{Ext}_A^m (N, L) \otimes \operatorname{Ext}_A^n (M, N) \longrightarrow \operatorname{Ext}_A^{m+n} (M, L)$$

Definition 2.3.2. *Let $(P_\bullet, \varepsilon)$ and $\left(Q_\bullet, \varepsilon'\right)$ be minimal projective resolutions of simple modules M and N respectively. Let $\eta \in \operatorname{Ext}_A^m(M, N)$ and $\xi \in \operatorname{Ext}_A^n(N, L)$. We have* *the following commutative diagram:*

$$\begin{tikzcd}
\cdots \arrow[r] & P_{m+n} \arrow[r] \arrow[d, "\iota_n"] & \cdots \arrow[r] & P_m \arrow[r] \arrow[d, "\iota_0"] \arrow[dr, "\eta"] & \cdots \arrow[r] & M \arrow[r] & 0 \\
\cdots \arrow[r] & Q_n \arrow[r] \arrow[d, "\xi"] & \cdots \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0 & \\
& L & & & & &
\end{tikzcd}$$

*where $\iota_0, ..., \iota_n$ denote successive liftings of η. Then we define the **Yoneda product** of ξ and η as*

$$\xi \cdot \eta = \xi \circ \iota_n.$$

If $\eta \in \operatorname{Ext}_A^m(M, N)$ and $\xi \in \operatorname{Ext}_A^n(R, L)$ and N and R are not isomorphic as A-modules, then we define $\xi \cdot \eta = 0$.

Proposition 2.6. *The Yoneda product is a well-defined associative bilinear product.*

Proof. See Carlson [Car96, pages 26-38] and [CTVEZ03, pages 61-64]. □

There are two important questions that we need to ask and answer before we go about trying to implement the Yoneda product into an algorithm. Do lifts always exist? If the lifts are not unique, then how do they affect computations in cohomology? The following proposition answers these two questions.

Proposition 2.7. *Suppose that M and N are simple A-modules with corresponding minimal projective resolutions $(P_\bullet, \varepsilon)$ and $\left(Q_\bullet, \varepsilon'\right)$ and that $\eta \in \operatorname{Ext}_A^m(M, N)$. We are in the following situation:*

$$\begin{tikzcd}
\cdots \arrow[r] & P_{m+2} \arrow[r, "\partial_{m+2}"] & P_{m+1} \arrow[r, "\partial_{m+1}"] & P_m \arrow[r, "\partial_m"] \arrow[dr, "\eta"] & \cdots \arrow[r] & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "\partial_2'"] & Q_1 \arrow[r, "\partial_1'"] & Q_0 \arrow[r, "\varepsilon'"] & N \arrow[r] & 0 & &
\end{tikzcd} \tag{2.3}$$

Then there exists a chain map $\{\iota_n\}_{n\in\mathbb{N}}$ which lifts η such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{m+2} \arrow[r, "\partial_{m+2}"] \arrow[d, "\iota_2"] & P_{m+1} \arrow[r, "\partial_{m+1}"] \arrow[d, "\iota_1"] & P_m \arrow[r, "\partial_m"] \arrow[d, "\iota_0"] \arrow[dr, "\eta"] & \cdots \arrow[r] & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "\partial_2^{'}"] & Q_1 \arrow[r, "\partial_1^{'}"] & Q_0 \arrow[r, "\varepsilon^{'}"] & N \arrow[r] & 0 & &
\end{tikzcd} \tag{2.4}$$

Moreover, any two such chain maps ι and $\iota^{'}$ that lift η are chain homotopic.

Proof. First we note that ι_0 exists because $\varepsilon^{'}$ is onto and P_m is a projective A-module. By induction we assume that we have constructed $\iota_k : P_{m+k} \to Q_k$ with $\iota_{k-1} \circ \partial_{m+k} = \partial^{'} \circ \iota_k$ for $k = 0, ..., n$. Then by the induction hypothesis we have

$$\partial_n^{'} \circ \iota_n \circ \partial_{m+n+1} = \iota_{n-1} \circ \partial_{m+n} \circ \partial_{m+n+1} = 0.$$

Thus $\iota_n \circ \partial_{m+n+1} : P_{m+n+1} \to Q_n$ has the property that $\partial_n^{'} \circ \iota_n \circ \partial_{m+n+1} = 0$. Now as the bottom row is also a projective resolution and thus exact, we have that

$$\iota_n \circ \partial_{m+n+1} \subseteq \partial_{n+1}^{'}\left(Q_{n+1}\right).$$

We therefore are in the following situation:

$$\begin{tikzcd}
P_{m+n+1} \arrow[d] \arrow[dr, "\iota_n \circ \partial_{m+n+1}"] & \\
Q_{n+1} \arrow[r, two heads] & \partial_{n+1}^{'}\left(Q_{n+1}\right)
\end{tikzcd}$$

As P_{m+n+1} is a projective A-module, we thus have a map $\iota_{n+1} : P_{m+n+1} \to Q_{n+1}$ as desired.

Now we show that any lift will do. Let $\iota^{'}$ be another chain map lifting η. It is true that φ_0 exists in (2.5) by the projectivity of P_m. Assume by induction that for $k = 0, ..., n$ that there exists $\varphi_k : P_{m+k} \to Q_{k+1}$ with $\iota_k - \iota_k^{'} = \partial_{k+1}^{'} \circ \varphi_k + \varphi_{k-1} \circ \partial_{m+k}$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_{m+2} \arrow[r, "\partial_{m+2}"] \arrow[d, shift right, "\iota_2^{'}"'] \arrow[d, shift left, "\iota_2"] & P_{m+1} \arrow[r, "\partial_{m+1}"] \arrow[dl, "\varphi_1"'] \arrow[d, shift right, "\iota_1^{'}"'] \arrow[d, shift left, "\iota_1"] & P_m \arrow[r, "\partial_m"] \arrow[dl, "\varphi_0"'] \arrow[d, shift right, "\iota_0^{'}"'] \arrow[d, shift left, "\iota_0"] \arrow[dr, "\eta"] & \cdots \arrow[r] & M \arrow[r] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "\partial_2^{'}"] & Q_1 \arrow[r, "\partial_1^{'}"] & Q_0 \arrow[r, "\varepsilon^{'}"] & N \arrow[r] & 0 &
\end{tikzcd} \tag{2.5}$$

Using the induction hypothesis and commutativity of the squares in the diagram we have that:

$$\begin{aligned}
\partial^{'}_{n+1} \circ \left(\iota_{n+1} - \iota^{'}_{n+1} - \varphi_n \circ \partial_{m+n+1}\right) &= \partial^{'}_{n+1} \circ \left(\iota_{n+1} - \iota^{'}_{n+1}\right) - \partial^{'}_{n+1} \circ \varphi_n \circ \partial_{m+n+1} \\
&= \left(\iota_n - \iota^{'}_n\right) \circ \partial_{m+n+1} - \partial^{'}_{n+1} \circ \varphi_n \circ \partial_{m+n+1} \\
&= \left(\partial^{'}_{n+1} \circ \varphi_n + \varphi_{n-1} \circ \partial_{m+n}\right) \circ \partial_{m+n+1} \\
&\qquad -\partial^{'}_{n+1} \circ \varphi_n \circ \partial_{m+n+1} \\
&= 0
\end{aligned}$$

As $\left(Q_\bullet, \varepsilon^{'}\right)$ is exact we have that

$$\left(\iota_{n+1} - \iota^{'}_{n+1} - \varphi_n \partial_{m+n+1}\right)\left(P_{m+n+1}\right) \subseteq \partial^{'}_{n+2}\left(Q_{n+2}\right)$$

and so there exists $\varphi_{n+1} : P_{m+n+1} \rightarrow Q_{n+2}$ with

$$\iota_{n+1} - \iota^{'}_{n+1} = \partial^{'}_n \varphi_{m+n+1} + \partial_{m+i} \varphi_{i-1}.$$

□

As we take the computational point of view of computing cohomology using projective resolutions, we must have a concrete way of constructing liftings. We have proven that lifts exist in Proposition 2.7. We compute the lifts in our program with the following algorithm.

Algorithm 2.3.1. *Lift Homomorphism Between Projective Modules*
We are in the following situation: $P = e_{p_1}A \dotplus \cdots \dotplus e_{p_l}A$, $Q = e_{q_1}A \dotplus \cdots \dotplus e_{q_m}A$, *and* $R = e_{r_1}A \dotplus \cdots \dotplus e_{r_n}A$ *We have homomorphisms (given as matrices)* $f : P \rightarrow Q$ $(l \times m)$ *and* $d : R \rightarrow Q$ $(n \times m)$, *such that the* $f(P) \subseteq d(R)$. *We calculate a lift* $\iota : P \rightarrow R$ *such that* $d \circ \iota = f$.

Input: P, Q, R *(as above), matrices* $f : P \rightarrow Q$ *and* $d : R \rightarrow Q$.

Output: *A homomorphism* ι *such that* $d \circ \iota = f$ *given as an* $l \times n$ *matrix* $\mathcal{M}$ *where each entry* $\mathcal{M}_{i,j}$ *gives the image of the idempotent* e_{p_i} *in* $e_{r_j}A$.

Initialize: $\mathcal{M} := 0_{l\times n}$ *(zero matrix with entries in* $\Bbbk$*).*
for i *from* 1 *to* l ***do***
  for j *from* 1 *to* n ***do***
    $\mathcal{S} := \{\gamma \in \mathrm{Basis}_{\Bbbk}(e_{p_i}B) : \tau(\gamma) = e_{r_j}\}$
    $\mathcal{N} := 0_{|S|\times m}$
    for t *from* 1 *to* $|S|$ ***do***
      $\mathcal{N}_{row\ t} := d(\gamma_t)$
    end for
    $v := f(0, ..., 0, e_{p_i}, 0, ..., 0)$ *with* e_{p_i} *in the* i^{th} *position.*
    Find x *such that* $x \cdot \mathcal{N} = v$
    $\mathcal{M}_{i,j} := x$
  end for
end for
return $\mathcal{M}$ *which is our required* ι*.*

Proposition 2.8. *The above algorithm 2.3.1 terminates and is correct.*

Proof. This is just a straight forward application of linear algebra. $\square$

We now define the Ext-algebra.

Definition 2.3.3. *Let* $S_1, ..., S_t$ *be the simple* A*-modules up to isomorphism. The* ***Ext-algebra*** $E(A)$ *(also called the* ***Yoneda algebra****) of* A *is:*

$$E(A) = \dotplus_{n=0}^{\infty} \dotplus_{i,j=1}^{t} \mathrm{Ext}_A^n(S_i, S_j)$$

where the multiplicative structure is given by the Yoneda product. If $\eta \in \mathrm{Ext}_A^n(S_i, S_j)$*, we say that the* ***degree*** *of* η *is* n*. The algebra* $E(A)$ *is a graded* $\Bbbk$*-algebra in a natural way given by the* n*.*

Lemma 2.9. *If* $\mathrm{Ext}_A^1(S_1, S_2) = 0$*, then every extension of* S_1 *by* S_2 *is split.*

Proof. For a proof see [Wei94, page 77]. $\square$

Assume that the algebra A is finite dimensional. Then $E(A)$ need not be finite dimensional, in fact $E(A)$ need not even be Noetherian. However, we shall see that for a group algebra, we always have finite generation. Let us also mention that the $E(A)$ is usually not commutative.

***Example* 2.3.1.** *Let $\Lambda = \Bbbk\Gamma/I$ where Γ is the quiver*

$$\begin{tikzcd} v_1 \arrow[r, shift left, "a"] & v_2 \arrow[l, shift left, "b"] \end{tikzcd}$$

and let $I = \langle aba, bab\rangle$. Let S_1 and S_2 denote the vertex simple modules and let $P_1 = e_{v_1}\Lambda$ and $P_2 = e_{v_2}\Lambda$ denote their projective covers. Let $\eta \in \operatorname{Ext}^1_\Lambda(S_1, S_2)$ and $\gamma \in \operatorname{Ext}^4_\Lambda(S_2, S_2)$ be nonzero. We compute $\eta \cdot \gamma$ and $\gamma \cdot \eta$ using projective resolutions. We have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\cdot a"] \arrow[d, equal, "\mathrm{id}"] & P_1 \arrow[r, "\cdot ab"] \arrow[d, "\cdot b"] & P_1 \arrow[r, "\cdot b"] \arrow[d, equal, "\mathrm{id}"] & P_2 \arrow[r, "\cdot ba"] \arrow[d, "\cdot a"] & P_2 \arrow[r, "\cdot a"] \arrow[d, equal, "\mathrm{id}"] \arrow[dr, "\eta"] & P_1 \arrow[r, "\varepsilon"] & S_1 \arrow[r] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r, "\cdot ba"] \arrow[d, "\gamma"] & P_2 \arrow[r, "\cdot a"] & P_1 \arrow[r, "\cdot ab"] & P_1 \arrow[r, "\cdot b"] & P_2 \arrow[r, "\varepsilon'"] & S_2 \arrow[r] & 0 & \\
 & S_2 & & & & & & &
\end{tikzcd}$$

where the top and bottom row are minimal projective resolutions of S_1 and S_2 respectively. The indicated maps are multiplication on the left by the images of the given arrows in the quotient Λ. More specifically we have that under $\cdot b$ we have the idempotent $e_{v_1} \in P_1$ maps to $b \cdot e_{v_1} = b$, etc. Here we take the identity map for ι_4 and therefore $\gamma \cdot \eta \neq 0$ in $\operatorname{Ext}^5_\Lambda(S_1, S_2)$ as we took $\gamma \neq 0$. It is clear from the definition of the Yoneda product that $\eta \cdot \gamma = 0$ as S_1 and S_2 are not isomorphic as Λ modules.

Definition 2.3.4. *Let $\Bbbk G$ be a group algebra and let $\Bbbk$ denote the trivial $\Bbbk G$-module. We define the* ***cohomology ring*** *of the group G as*

$$H^*(G, \Bbbk) = Ext^*_{\Bbbk G}(\Bbbk, \Bbbk) = \dotplus_{k=0}^{\infty} \operatorname{Ext}^k_{\Bbbk G}(\Bbbk, \Bbbk),$$

a subring of the Ext*-algebra of* $\Bbbk G$*.*

Proposition 2.10. $H^*(G, \Bbbk) = \operatorname{Ext}^*_{\Bbbk G}(\Bbbk, \Bbbk)$ *is a graded commutative ring (i.e.* $xy = (-1)^{\deg x \cdot \deg y} yx$*) and in the case when* $\operatorname{char} \Bbbk = 2$*, we have a commutative ring.*

Proof. See Carlson [Car96, page 38]. □

The cohomology ring for a group has some important properties and interpretations. By taking the trivial $\Bbbk G$-module, it focuses attention on the group G itself, that is, group cohomology can be used to reflect the internal structure of G such as its p-rank. If M is a $\Bbbk G$-module, the second cohomology group $H^2(G, M)$ is in one-to-one correspondence with the set of equivalence classes of extensions of G by M up an equivalence relation. Since homology theory is rooted in topology, it can also be used to study the possible ways a group can act on spaces or other sets with some structure. An example of the application of computing $H^*(G, \Bbbk)$ was the proof (due to P. Smith [Smi44]) that if any finite group acts freely on a sphere then all of its abelian subgroups must be periodic. The work of D. Quillen, J. Alperin, L. Evens, J. Carlson, and D. Benson connects the cohomology ring of a finite group with coefficients in a finite field to the structure of modular representations of G. The theory of this is discussed in Benson [Ben98b, Ben98a].

We do not wish to go further into the interpretations and properties of $H^*(G, \Bbbk)$. Our goal is to supply the techniques and programs needed to provide examples to better understand the theory.

2.4 Computing $H^*(G, \Bbbk)$ and $E(\Bbbk G)$

When we consider computing the Ext-algebra, we do this on a block by block basis. Recall that if P_i and P_j are in different blocks, that there are no nontrivial homomorphisms between them. Thus for any simple module S_i and S_j corresponding to P_i and P_j in **different blocks** we have that $\operatorname{Ext}^n_{\Bbbk G}(S_i, S_j) = 0$. In our computations and results we provide, we compute the Ext-algebra for the principal block. However, the techniques that we have provided work for all blocks of a group algebra $\Bbbk G$.

As noted before, to compute $E(\Bbbk G)$ and $H^*(G, \Bbbk)$ we would prefer to compute in a smaller algebra with the same homological properties. The theorem that allows us to work in a Morita equivalent ring is the following.

Theorem 2.11. *Assume that for two finite dimensional algebras A and B we have that A is Morita equivalent to B given by the functors $F : A \to B$ and $G : B \to A$. Let S_i and S_i' denote the respective simple modules. Then*

$$\dot{+}_{n=0}^{\infty} \dot{+}_{i,j} \operatorname{Ext}_A^n(S_i, S_j) \cong \dot{+}_{n=0}^{\infty} \dot{+}_{i,j} \operatorname{Ext}_B^n\left(S_i^{'}, S_j^{'}\right)$$

as algebras.

Proof. Let $S_1, ..., S_r$ be the simple A-modules with corresponding projective covers $P_1, ..., P_r$ by Theorem 1.25 on page 41. Then by Lemma 1.31 on page 49 we know that $F(S_1), ..., F(S_r)$ and $F(P_1), ..., F(P_r)$ are the simples and corresponding PIMs for B. Let $(P_\bullet, \varepsilon)$ be a minimal projective resolution for a simple S_i. Then we know that $(F(P_\bullet), F(\varepsilon))$ is a projective resolution for $F(S_i)$. As $(P_\bullet, \varepsilon)$ is minimal we know that $\operatorname{Im}\partial_n \subseteq \operatorname{Rad} P_{n-1}$ and so $F(\operatorname{Im}\partial_n) \subseteq F(\operatorname{Rad} P_{n-1}) = \operatorname{Rad} F(P_{n-1})$. Thus $(F(P_\bullet), F(\varepsilon))$ is a minimal projective resolution. Consider any $\eta \in \operatorname{Ext}_A^n(S_i, S_j)$. By Proposition 2.5 on page 62 we know that $\eta \in \operatorname{Hom}_A\left(P_n^{'}, S_j\right)$. But by Proposition 1.33 on page 51 we have that

$$\operatorname{Hom}_A\left(P_n^{'}, S_j\right) \cong \operatorname{Hom}_B\left(F\left(P_n^{'}\right), F(S_j)\right).$$

A further investigation shows the multiplicative structure is also compatible. Therefore as this is true for all simples S_i and all n, we have isomorphic Ext-algebras.

For a complete proof see McCarthy [McC88, pages 211-215]. $\square$

Our ultimate goal is to compute the Ext-algebra and cohomology ring for a group algebra $\Bbbk G$. Theorem 2.11 allows us to make this calculation easier by working in the Morita equivalent basic algebra B. We shall denote the image of the trivial $\Bbbk G$-module $\Bbbk$ under the Morita equivalence F (with inverse G) by $F(\Bbbk) := \Bbbk_B$. Therefore as $E(\Bbbk G) \cong E(B)$ and $H^*(G, \Bbbk) = \operatorname{Ext}^*_{\Bbbk G}(\Bbbk, \Bbbk) \cong \operatorname{Ext}^*_B(\Bbbk_B, \Bbbk_B)$, we have designed and implemented our algorithm to work for basic algebras.

The first step in computing the Ext-algebra for a basic algebra B is to compute the projective resolutions of the simple B-modules $S_1, ..., S_t$ to a given degree n. After computing the projective resolutions we will have determined the $\Bbbk$-dimensions of the vector spaces $\operatorname{Ext}^k_B(S_i, S_j)$, $1 \leq k \leq n$. But we also want the multiplicative structure.

We determine the multiplicative structure by first finding a minimal set of generators for $\dotplus_{k=0}^{n} \dotplus_{i,j=1}^{t} \operatorname{Ext}^k_B(S_i, S_j)$ and then determining all possible products in these generators up to degree n.

Let $\mathcal{B}_{i,j,k}$ be a basis for the $\Bbbk$-vector space $\operatorname{Ext}^k_B(S_j, S_j)$ and $\mathcal{B}^{\text{Yon}}_{i,j,k}$ a basis for the corresponding zero-dimensional subspace of $\operatorname{Ext}^k_B(S_i, S_j)$ for $1 \leq k \leq n$ and $i, j = 1, ..., t$. We compute a minimal generating set as follows. Let $\mathfrak{G} := \emptyset$ be our set of generators. Let $m > 0$ be the smallest integer such that there are i_0 and j_0 such that $\dim_{\Bbbk} \operatorname{Ext}^m_B(S_{i_0}, S_{j_0}) = r > 0$. Let $\eta_{i_0,j_0,m_1}, ..., \eta_{i_0,j_0,m_r} \in \mathcal{B}_{i_0,j_0,m}$. For $\eta_{i_0,j_0,m_1}, ..., \eta_{i_0,j_0,m_r} \in \mathcal{B}_{i_0,j_0,m}$, if $\eta_{i_0,j_0,m_l} \notin \operatorname{Span} \mathcal{B}^{\text{Yon}}_{i_0,j_0,m}$ then add η_{i_0,j_0,m_l} to the generating set $\mathfrak{G}$ and also to $\mathcal{B}^{\text{Yon}}_{i_0,j_0,m}$. We then lift each of the η_{i_0,j_0,m_l} for $l = 1, ..., r$ as in Figure 2.1:

$$
\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r, "\partial_n"] \arrow[d, "\iota_{n-m_l}"] & \cdots \arrow[r] & P_{m+2} \arrow[r, "\partial_{m+2}"] \arrow[d, "\iota_2"] & P_{m+1} \arrow[r, "\partial_{m+1}"] \arrow[d, "\iota_1"] & P_m \arrow[r] \arrow[d, "\iota_0"] \arrow[dr, "\eta_{i_0,j_0,m_l}"] & \cdots \arrow[r, "\epsilon"] & S_{i_0} \arrow[r] & 0 \\
\cdots \arrow[r] & P'_{n-r} \arrow[r, "\partial'_{n-r}"] \arrow[d, "\eta_{j_0,k,n-m_l}"] & \cdots \arrow[r] & P'_2 \arrow[r, "\partial'_2"] \arrow[d, "\eta_{j_0,k,2}"] & P'_1 \arrow[r, "\partial'_1"] \arrow[d, "\eta_{j_0,k,1}"] & P'_0 \arrow[r, "\epsilon'"] & S_{j_0} \arrow[r] & 0 & \\
& S_k & & S_k & S_k & & & &
\end{tikzcd}
$$

FIGURE 2.1. Standard Lifting of a Generator

In Figure 2.1 we have $\eta_{j_0,k,s} \circ \eta_{i_0,j_0,m_l} = \eta_{j_0,k,s} \circ \iota_s$ for $1 \leq s \leq n-r$. We compute $\eta_{j_0,k,s} \circ \eta_{i_0,j_0,m_l}$ for all $\eta_{j_0,k,s} \in \mathcal{B}_{j_0,k,s}$ and all $k = 1, ..., t$. We then add all of these products $\eta_{j_0,k,s} \circ \eta_{i_0,j_0,m}$ to $\mathcal{B}^{\text{Yon}}_{i_0,k,s+m}$.

We then proceed to the next i_1, j_1 such that $\dim_{\Bbbk} \operatorname{Ext}^m_B(S_{i_1}, S_{j_1}) > 0$. We consider all $\eta_{i_1,j_1,m} \in \mathcal{B}_{i_1,j_1,m}$ such that $\eta_{i_1,j_1,m} \notin \operatorname{Span} \mathcal{B}^{Yon}_{i_1,j_1,m}$. We then repeat the above lifting procedure. We do this for all i_α and j_β in degree m. We then proceed to degree $m+1$ and repeat until we eventually get to degree n.

We now make the above description into an algorithm that we implement into `GAP`. The algorithm finds a minimal generating set for the Ext-algebra $E(B)$ up to degree n, i.e. a generating set for $\dotplus_{k=0}^{n} \dotplus_{i,j} \operatorname{Ext}^k_B(S_i, S_j)$.

Algorithm 2.4.1. *Compute Minimal Generators*

Input: *A basic algebra B and desired degree of computation n.*

Output: *Minimal set of generators of* Ext*-algebra to degree n, $\dotplus_{k=0}^{n} \dotplus_{i,j} \operatorname{Ext}^k_B(S_i, S_j)$.*

Initialize the generators, $\mathfrak{G} := \emptyset$.

$N :=$ Number of Simple B-modules, S_i.

for** i from 1 to N **do

  Compute minimal projective resolution for S_i to degree n using Algorithm 2.2.1.

end for

Initialize $\mathcal{B}^{\text{Yon}}_{i,j,k} := \emptyset$, $1 \leq i,j \leq N$, $1 \leq k \leq n$, the basis for the space of Yoneda products of the generators and $\operatorname{Basis}_{\Bbbk}\left(\operatorname{Ext}^k_B(S_i, S_j)\right) := \mathcal{B}_{i,j,k}$.

$D_{i,j,k} := \dim_{\Bbbk} \operatorname{Ext}^k_B(S_i, S_j)$, $1 \leq i,j \leq N$, $1 \leq k \leq n$.

for** k from 1 to n **do

  for** i from 1 to N **do

    for** j from 1 to N **do

      if** $D_{i,j,k} \neq 0$ **then

        for** $\eta \in \mathcal{B}_{i,j,k}$ **do

          if** $\eta \notin \operatorname{Span}_{\Bbbk}\left(\mathcal{B}^{\text{Yon}}_{i,j,k}\right)$ **then

Add η *to* $\mathfrak{G}$
Add η *to* $\mathcal{B}^{\text{Yon}}_{i,j,k}$
for m *from* 1 *to* $n-k$ **do**
Compute lift ι_m *for* η *with projective resolutions of* S_i *and* S_j *using Algorithm 2.3.1 as in Figure 2.1.*
for r *from* 1 *to* N **do**
for $\gamma \in \mathcal{B}_{j,r,m}$ **do**
Compute $\gamma \cdot \iota_m$
if $\gamma \cdot \iota_m \notin \operatorname{Span}_{\Bbbk} \mathcal{B}^{\text{Yon}}_{i,r,m+k}$ **then**
Add $\gamma \cdot \iota_m$ *to* $\mathcal{B}^{\text{Yon}}_{i,r,m+k}$
end if
end for
end for
end for
end if
end for
end if
end for
end for
end for
return *Generators* $\mathfrak{G}$ *for* $E(B)$.

Proposition 2.12. *Algorithm 2.4.1 produces a minimal generating set for* $\dotplus_{k=0}^{n} \dotplus_{i,j} \operatorname{Ext}^k_B(S_i, S_j)$.

Proof. It is clear from the construction that we produce a generating set. What we must prove is that we have found a minimal generating set. We proceed by induction on the degree k. It is clear that we have a minimal generating set up to $k = 1$ as there is no way to write a generator in degree 1 as the product of two other positive degree generators. Now assume that we have a minimal generating set for $k = 1, ..., n-1$. Assume that $\eta \in \mathrm{Ext}_B^n(S_i, S_j)$ is a generator that we have found of degree n. Then we know by construction that η cannot be written as a linear combination of generators and basis elements of lower degree. Therefore η is a necessary generator and is part of the minimal set. $\square$

Now that we have a minimal generating set for $\dotplus_{k=0}^{n} \dotplus_{i,j} \mathrm{Ext}_B^k(S_i, S_j)$, we would like to rewrite the basis for $\dotplus_{k=0}^{n} \dotplus_{i,j} \mathrm{Ext}_B^k(S_i, S_j)$ in terms of the generators $\{\eta_1, ..., \eta_r\}$. This will then allow us to easily find the ideal of relations satisfied by the generators for the algebra and also compute a Gröbner basis $\mathcal{G}$ for the ideal of generators relations such that $\dotplus_{k=0}^{n} \dotplus_{i,j} \mathrm{Ext}_B^k(S_i, S_j) \cong \Bbbk\langle \eta_1, ..., \eta_r\rangle / \langle \mathcal{G} \rangle$. We describe the process of rewriting the basis in the following algorithm.

Algorithm 2.4.2. *Spinning*

Assume that the $\Bbbk$-algebra $E(B)$ up to degree n is given by generators $\{\eta_1, ..., \eta_r\}$ and that the graded vector space is given as $V = \dotplus_{k=0}^{n} \dotplus_{i,j} Ext_B^k(S_i, S_j)$.

Input: *Algebra generators $\{\eta_1, ..., \eta_r\}$ of $E(B)$ ordered by degree.*

Output: *A graded $\Bbbk$-basis $\mathcal{B}$ for V given as products in the generators $\{\eta_1, ..., \eta_r\}$.*

```
Initialize B = {η_1, ..., η_r}
for k from 1 to degree n of computation do
    for b ∈ B do
        for i from 1 to r do
            v := η_i · b (Yoneda Product)
            if degree v = k then
                if v ∉ Span_k B then
                    Append v to the end of the list B
                end if
            end if
        end for
```

    end for
end for
return $\mathcal{B}$

Lemma 2.13. *The above algorithm terminates and is correct.*

Proof. Algorithm 2.4.2 terminates since we are only considering the basis up to a finite n. By construction $\mathcal{B}$ is linearly independent over $\Bbbk$, $\mathcal{B} \subseteq E(B)$, and $\{\eta_1, ..., \eta_r\}$ is a generating set. $\square$

We now have a new basis for the Ext-algebra as a graded $\Bbbk$-vector space and we also have a record of each of the products in the generators. The next step is to see what relations we have between the generators and would like to describe the ideal I of these relations in the form of a Gröbner basis.

Algorithm 2.4.3. *Compute Gröbner Basis for Relations of $E(B)$*
We wish to compute the relations in the generators for $E(B)$ and present them as a Gröbner basis $\mathcal{G}$.

Input: *The generators of the* Ext-*algebra $E(B)$ up to a given degree n.*
Output: *A Gröbner basis for the relations ideal of $E(A)$.*
Rewrite Basis in terms of Generators using Spinning Algorithm 2.4.2
Compute Relations $\mathcal{G}$ by using Alternative Gröbner Basis Algorithm 3.1.3 in chapter 3.
return $\mathcal{G}$

For an expository description of the implementation of the computation of the Ext-algebra including technical remarks and examples, see Chapter 4.

2.4.1 The Quiver of B and $E(B)$

Definition 2.4.1. *Let $S_i \cong e_iB/e_i\,\mathrm{Rad}(B)$ for $i = 1, ..., r$, be representatives for the isomorphism classes of simple A-modules. The **Ext-quiver** of B, $Q(B)$, is the quiver* *with vertices x_i corresponding to S_i, and* $\dim_{\Bbbk} \operatorname{Ext}^1_B(S_i, S_j)$ *arrows from x_i to x_j.*

Lemma 2.14. *Let u, v be vertices in a special quiver with relations (Γ, I). Let $\Lambda = \Bbbk\Gamma/I$. Then* $\dim_{\Bbbk} \operatorname{Ext}^1_\Lambda(S_v, S_u)$ *is equal to the number of arrows from v to u in Γ.*

Proof. Denote by Γ_1^v the set of arrows in Γ with origin v. For $a \in \Gamma_1^v$, left multiplication by a is a homomorphism $\tau(a)\Lambda \to v\Lambda$. This gives rise to the exact sequence

$$\oplus_{a \in \Gamma_1^v} \tau(a)\Lambda \xrightarrow{\partial_1} e_v\Lambda \longrightarrow S_v \longrightarrow 0, \tag{2.6}$$

with $\operatorname{Im}(\partial_1) = \operatorname{Rad}(e_v\Lambda)$. So (2.6) is the start of a projective resolution of S_v, minimal at degree 0. Since $I \subseteq J^2$, it follows that $\operatorname{Ker}(\partial_1) \subseteq \operatorname{Rad}\left(\bigoplus_a \tau(a)\Lambda\right)$, and so the resolution is also minimal at degree 1. The degree 1 projective indecomposable module P in the minimal resolution therefore has $P/\operatorname{Rad} P = \bigoplus_{a \in \Gamma_1^v} S_{\tau(a)}$. Thus the result follows. $\square$

The basic algebra B that we are given can be constructed as a path algebra modulo an ideal of relations. We can visually picture this via its Ext-quiver. In the Ext-quiver we know that the vertices are the idempotents in the algebra corresponding to the simple modules S_i and the arrows from S_i to S_j represent elements in $\operatorname{Ext}^1_B(S_i, S_j)$. We now define a way to give a pictorial description of $E(B)$.

Definition 2.4.2. *Let $E(B)$ be the* Ext*-algebra of a finite dimensional algebra B. We define the **quiver of** $E(B)$ denoted $Q(E(B))$ as follows: We make a vertex v_i to represent each of the spaces* $\operatorname{Ext}^0_A(S_i, S_i)$*. We then make arrows from vertex v_i to v_j for each generator* $\eta_{i,j,k} \in \operatorname{Ext}^k_A(S_i, S_j)$.

We note that we are able to read off the Ext-quiver of B from the quiver of $E(B)$ by keeping the vertices and the degree 1 arrows. The quiver $Q(E(B))$ for $E(B)$ therefore contains $Q(B)$ as a subquiver and we can get $Q(E(B))$ from $Q(B)$ just by drawing more arrows. We denote the arrows in $Q(E(B))$ from S_i to S_j denoted by $\eta_{i,j,k}$ where k is the degree of the generator.

***Example* 2.4.1.** *We recall that the* Ext*-quiver for* $\mathbb{F}_2 S_4$ *is:*

$$\begin{tikzcd} 1a \arrow[loop left] \arrow[r, shift left] & 2a \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

The quiver for the Ext*-algebra* $E(\mathbb{F}_2 S_4)$ *is given in Figure 2.2.*

$$\begin{tikzcd} 1a \arrow[loop left, "\eta_{1,1,1}"] \arrow[loop above, "\eta_{1,1,2}"] \arrow[r, shift left, "\eta_{1,2,1}"] & 2a \arrow[l, shift left, "\eta_{2,1,1}"] \arrow[loop right, "\eta_{2,2,1}"] \end{tikzcd}$$

FIGURE 2.2. Quiver of Ext-algebra of $\mathbb{F}_2 S_4$

2.5 Finite Generation of $H^*(G, \Bbbk)$ and $E(\Bbbk G)$

In the case of a group algebra $\Bbbk G$ we know that both the cohomology ring and Ext-algebra are infinite dimensional graded vector spaces. Evens has shown that the cohomology ring and the Ext-algebra are finitely generated as $\Bbbk$-algebras. Therefore, one goal is to describe this noncommutative infinite dimensional algebra in terms of a finite set of generators and the relations satisfied by the generators.

Theorem 2.15. *(Evens, Venkov) If G is a finite group then the cohomology ring $H^*(G, \Bbbk)$ is a finitely generated $\Bbbk$-algebra.*

Proof. The proof can be found in Evens [Eve61]. □

Corollary 2.16. *The* Ext*-algebra $E(\Bbbk G)$ is finitely generated as a $\Bbbk$-algebra.*

Proof. See Benson [Ben98b, page 127] □

An essential question in the course of a computer calculation of the cohomology ring and Ext-algebra concerns what degrees a minimal set of generators and relations should lie in. For the computer calculations in the appendix, the projective resolutions of the simple $\Bbbk G$-modules are only computed out to 20 degrees and for the Ext-algebra we have computed many only up to $n = 12$ or less. It is a problem to know exactly when we have found all of the generators and relations to determine a presentation of the Ext-algebra or cohomology ring. Carlson has a technique for cohomology that relies on restrictions to subgroups to find if he has found enough generators and relations [Car01, CTVEZ03]. The technique uses restrictions of the group to certain subgroups and he is able to prove that he has found enough generators by using this technique. However, in our case, what we have gained in computational power for large groups by passing to the basic algebra, we have lost in terms of group structure information.

Chapter 3

Noncommutative Gröbner Bases

Thus far we have provided a way of constructing a projective resolution that relies only on techniques from linear algebra. An alternate approach to computing projective resolutions will use Gröbner basis theory. We wish to develop a theory for Gröbner bases over path algebras, which are generally noncommutative algebras. The reason we use this approach is motivated by the following theorem.

Theorem 3.1. *(Gabriel) Suppose B is a finite dimensional basic algebra over a splitting field $\mathbb{k}$ and let $\Gamma = Q(B)$ be its* Ext*-quiver. Then there is a $\mathbb{k}$-algebra epimorphism $\Phi : \mathbb{k}\Gamma \twoheadrightarrow B$ where the kernel of Φ is contained in the ideal generated by the paths of length* 2*.*

Proof. For a proof see Benson [Ben98a, page 103]. $\square$

3.1 Noncommutative Gröbner Bases

For computational purposes, we do not work in the group algebra $\mathbb{k}G$. Instead we prefer the Morita equivalent basic algebra B. From Theorem 3.1 we know there is an ideal $I = \operatorname{Ker}\phi$ such that $B \cong \mathbb{k}\Gamma/I$. To study the quotient algebra $\mathbb{k}\Gamma/I$, we would like to have a good method for working with the equivalence classes of the form $f + I$. This means we have to be able to compute the normal form for elements of equivalence classes efficiently. This is where the concept of a Gröbner basis appears. A Gröbner basis, moreover, provides a way of computing projective resolutions which originates in [AG87] and [FGKK93]. First we present the concept of a Gröbner basis in a noncommutative setting. Then we discuss the more specific setting of a finite dimensional algebra $B \cong \Bbbk\Gamma/I$.

Let $\Bbbk$ be an arbitrary field and A an arbitrary $\Bbbk$-algebra with $\mathcal{B}$ a $\Bbbk$-basis of A. For example we may consider the $\Bbbk$-algebra with basis $\mathcal{B}$ consisting of all monomials in the indeterminates $x_1, ..., x_n$. In other words, we could consider the ring of polynomials $\Bbbk\langle x_1, ..., x_n\rangle$ in noncommutative indeterminates, i.e. $x_i x_j \neq x_j x_i$ for $i \neq j$. We will be considering the (two-sided) ideals in A. A good introduction to Gröbner basis theory in the commutative setting can be found in [Frö97, CLO97].

Definition 3.1.1. *A basis $\mathcal{B}$ for a $\Bbbk$-algebra A is said to be a **multiplicative basis** if for all b_i, $b_j \in \mathcal{B}$ we have $b_i \cdot b_j \in \mathcal{B}$ or $b_i \cdot b_j = 0$.*

Definition 3.1.2. *Given a basis $\mathcal{B}$, we say that $>$ is a **well-order** on $\mathcal{B}$ if $>$ is a total-order such that every nonempty subset of $\mathcal{B}$ has a smallest element.*

Definition 3.1.3. *Let $\mathcal{B}$ be a basis for a $\Bbbk$-algebra A. We say that $>$ is an **admissible order** on a multiplicative basis $\mathcal{B}$ if $>$ is a well order and for all $p, q, r, s \in \mathcal{B}$ we have*

1. *$p < q \implies spr < sqr$ if both $spr \neq 0$ and $sqr \neq 0$.*
2. *$p = qr \implies p > q$ and $p > r$.*

***Example* 3.1.1.** *Consider the* Ext*-quiver Γ for the basic algebra $B \cong_{Morita} \Bbbk S_4$ over $\mathbb{F}_2$. The two vertices are labeled 1a and 2a and there are four arrows, a,b,c, and d. There are two simple S_4 modules which are represented by the vertices 1a and 2a. The* Ext*-quiver is given in Figure 3.1 We give an admissible order on the basis of the path algebra $\mathbb{F}_2\Gamma$ as follows:*

$$v_1 < v_2 < a < b < c < d < aa < ac < ba < bc < cb < cd < db < dd < \cdots \tag{3.1}$$

The ordering that we will use throughout this dissertation is the admissible order commonly referred to as the length-lexicographic ordering. We define this ordering as follows.

$$\begin{tikzcd} 1a \arrow[loop left, "a"] \arrow[r, shift left, "c"] & 2a \arrow[l, shift left, "b"] \arrow[loop right, "d"] \end{tikzcd}$$

FIGURE 3.1. Ext-Quiver of Basic Algebra of $\mathbb{F}_2 S_4$

Definition 3.1.4. *Let $\Bbbk\Gamma$ be a path algebra with multiplicative basis $\mathcal{B}$. Pick a total ordering on the set of arrows and vertices in the quiver Γ. The **length-lexicographic ordering** on $\mathcal{B}$ is then defined as follows: $b_1 \leq b_2$ if*

1. $\mathrm{length}(b_1) < \mathrm{length}(b_2)$, *or*

2. $\mathrm{length}(b_1) = \mathrm{length}(b_2)$ *and $b_1 \leq b_2$ lexicographically.*

***Example* 3.1.2.** *Take the polynomial ring $\Bbbk[x, y]$ in two indeterminates with $x > y$ ordered length-lexicographically. Then $x^4y^3 > x^3y^4$ and $xy^3 < y^5$. Also, 3.1 in example 3.1.1 is an example of the length-lexicographic order.*

Definition 3.1.5. *Let $x = \sum_{i \in \mathcal{I}} \alpha_i b_i$, where $\alpha_i \in \Bbbk$, $b_i \in \mathcal{B}$ and only finitely many of the α_i are nonzero. We say that b_i is in the **support** of x if $\alpha_i \neq 0$. Denote this by* $\mathrm{supp}(x)$.

The notion of a largest basis element is necessary so we can define a leading term. Thus we define the Tip of an element $x \in A$ as follows (in the commutative case that most people are familiar with, Tip is often called leading term or head term).

Definition 3.1.6. *If $\mathcal{B} = \{b_i\}_{i \in \mathcal{I}}$ is a basis of our $\Bbbk$-algebra A and $>$ is a well-order on $\mathcal{B}$, then if $x = \sum_{i \in \mathcal{I}} \alpha_i b_i$ is a nonzero element of A, we say b_i is the **tip** of x if b_i is in the support of x and $b_i \geq b_j$ for all b_j in the support of x. We will denote this by* $\mathrm{Tip}\,(x)$. *We denote the coefficient of a tip as* $\mathrm{CTip}(x)$.

Definition 3.1.7. *If X is a subset of A with basis $\mathcal{B}$ we let*

$$\mathrm{Tip}\,(X) = \{b \in \mathcal{B} : b = \mathrm{Tip}\,(x) \text{ for some nonzero } x \in X\}.$$

We use $\mathrm{NonTip}(X)$ *to denote the set* $\mathcal{B}\backslash \mathrm{Tip}(X)$*. So both* $\mathrm{Tip}(X)$ *and* $\mathrm{NonTip}(X)$ *are subsets of our fixed basis* $\mathcal{B}$*. Both sets are* ***dependent*** *on the choice of admissible-order on* $\mathcal{B}$*.*

So whenever we write down $\mathrm{Tip}(X)$ or $\mathrm{NonTip}(X)$ it is assumed that this includes an admissible order $>$. For the rest of the thesis, we fix the admissible order $>$ as the length-lexicographic ordering.

Lemma 3.2. *Given an ideal* I *in a path algebra* $\Bbbk\Gamma$*, the following are properties of* $\mathrm{NonTip}\,(I)$*.*

1. *The cosets* $\{f + I : f \in \mathrm{NonTip}(I)\}$ *form a* $\Bbbk$*-basis for* $\Lambda = \Bbbk\Gamma/I$*.*
2. *Each coset of* I *in* $\Bbbk\Gamma$ *contains exactly one member of the span of* $\mathrm{NonTip}(I)$*.*
3. *The coset representative is the unique element of the coset with the smallest support.*

Proof. For a proof see D. Green [Gre97, page 20] $\square$

Definition 3.1.8. *For* $f \in \Bbbk\Gamma$*, denote by* $N_I\,(f)$ *the unique smallest support element of the coset* $f + I$*. This is the* ***standard coset representative*** *of* $f + I$*. The previous lemma 3.2 ensures that this definition makes sense, and that* $N_I\,(f)$ *is also the unique element of* $f + I$ *in the* $\Bbbk$*-span of* $\mathrm{NonTip}\,(I)$*.*

The tip and nontip sets give us a way to decompose an algebra as follows:

Theorem 3.3. *Let* A *be a* $\Bbbk$*-algebra with basis* $\mathcal{B}$*. Let* $>$ *be a well-order on* $\mathcal{B}$*. Suppose that* I *is an ideal in* A *. Then* $A = I \oplus \mathrm{Span}_{\Bbbk}\,(\mathrm{NonTip}\,(I))$*, as* $\Bbbk$*-vector spaces.*

Proof. A proof is found in [Gre99]. $\square$

Every nonzero $x \in A$ can be written uniquely as $i_x + N(x)$, where $i_x \in I$ and $N(x) \in \mathrm{Span}(\mathrm{NonTip}(I))$. We call $N(x)$ the **normal form** of x.

Now we define a Gröbner basis $\mathcal{G}$ for an ideal I in A, a $\Bbbk$-algebra with multiplicative basis $\mathcal{B}$ and admissible order $>$.

Definition 3.1.9. *We say that a set $\mathcal{G} \subseteq I$ is a **Gröbner basis** for I with respect to an admissible ordering $>$ if $\langle \mathrm{Tip}(\mathcal{G}) \rangle = \langle \mathrm{Tip}(I) \rangle$, the ideal generated by the tips of $\mathcal{G}$ is the same as the ideal generated by the tips of the ideal I.*

Gröbner bases can also be thought of in terms of division. We first define what we mean by division in a noncommutative setting.

Definition 3.1.10. *Given $x, y \in A$ with basis $\mathcal{B}$, we say that x **divides** y, denoted $x \mid y$, if there exist $p, q \in \mathcal{B}$ such that $pxq = y$.*

We now can state a proposition that gives another interpretation of Gröbner bases in terms of division.

Proposition 3.4. *Let I be an ideal in a $\Bbbk$-algebra A. Given an admissible ordering $>$, if for every $b \in \mathrm{Tip}(I)$ there is some $g \in \mathcal{G}$ such that $\mathrm{Tip}(g)$ divides b then $\mathcal{G}$ is a Gröbner basis for I.*

Proof. Assume that $\langle \mathrm{Tip}(\mathcal{G}) \rangle = \langle \mathrm{Tip}(I) \rangle$. Let $b \in \mathrm{Tip}(I)$. Therefore $b \in \langle \mathrm{Tip}(I) \rangle$ and thus $b \in \langle \mathrm{Tip}(\mathcal{G}) \rangle$. So $b = r_1 g_1 s_1 + \cdots + r_n g_n s_n$ with $g_i \in \mathcal{G}$ and $r_i, s_i \in R$. But b is a basis element and thus is monomial. Thus $r_i = 0$ for all but one i. Without loss of generality, let $r_1, s_1 \neq 0$. Then as we have a multiplicative basis, r_1 and $s_1 \in \mathcal{B}$. Conversely, assume for every $b \in \mathrm{Tip}(I)$ there is some $g \in \mathcal{G}$ such that $\mathrm{Tip}(g)$ divides b. Let $g \in \langle \mathrm{Tip}(\mathcal{G}) \rangle$, $g = r_1 \mathrm{Tip}(g_1) s_1 + \cdots + r_n \mathrm{Tip}(g_n) s_n = r_1 p_1 g_1 q_1 s_1 + \cdots + r_n p_n g_n q_n s_n$ and is thus in $\langle \mathrm{Tip}(I) \rangle$. Now let $t \in \langle \mathrm{Tip}(I) \rangle$. Then $t = r_1 b_1 s_1 + \cdots + r_n b_n s_n$ where b_i are basis elements in $\mathrm{Tip}(I)$. Then $t = r_1 p_1 b_1 q_1 s_1 + \cdots + r_n p_n b_n q_n s_n$ and as each $p_i b_i q_i = g_i \in \mathcal{G}$, then we have $\langle \mathrm{Tip}(I) \rangle \subseteq \langle \mathrm{Tip}(\mathcal{G}) \rangle$ and thus we have shown that $\langle \mathrm{Tip}(I) \rangle = \langle \mathrm{Tip}(\mathcal{G}) \rangle$. $\square$

An important fact about Gröbner bases to notice is that if a set $\mathcal{G}$ is a Gröbner basis for an ideal I then $\mathcal{G}$ generates I.

Lemma 3.5. *If $\mathcal{G}$ is a Gröbner basis for an ideal I then $\mathcal{G}$ generates I.*

Proof. By contradiction, suppose that $\mathcal{G}$ does not generate I. Let $x \in I \backslash \langle \mathcal{G} \rangle$ be such that $\mathrm{Tip}(x)$ is as small as possible (in the ordering). Then, since $\mathcal{G}$ is a Gröbner basis for I, there is some $g \in \mathcal{G}$ such that $\mathrm{Tip}\,(g) \,|\, \mathrm{Tip}\,(x)$. Thus $\mathrm{Tip}(x) = b\,\mathrm{Tip}(g)c$ for some $b, c \in \mathcal{B}$. If α is the coefficient of $\mathrm{Tip}(x)$ in x then consider $y = x - \alpha bgc$. By construction, $\mathrm{Tip}(y) < \mathrm{Tip}(x)$ and $y \in I \backslash \langle \mathcal{G} \rangle$. This contradicts the choice of x and we have shown that $\mathcal{G}$ generates I. □

As in the commutative case, the division of an element $y \in A$ by an ordered set of elements $X = \{f_1, ..., f_n\}$ of A is important. We need to emphasize that the order of the elements affects the outcome of the division algorithm. There is a division algorithm in the noncommutative setting as in the commutative setting.

Algorithm 3.1.1. *Division Algorithm*

Input: *An ordered set of polynomials $X = \{f_1, ..., f_n\}$ in A, a polynomial $y \in A$, and an admissible order $>$.*

Output: *The remainder r of the division of y by the set X.*

```
Initialize: m_1 := 0, ..., m_n := 0, r := 0, z := y, DIVOCCUR:=False
while z ≠ 0 and DIVOCCUR==False do
    for i from 1 to n do
        if Tip(z) = u Tip(f_i) v for u, v ∈ B then
            m_i := m_i + 1
            u_{i,m_i} := [CTip(z)/ CTip(x_i)] u (left most division)
            v_{i,m_i} := v
            z := z − [CTip(z)/ CTip(x_i) u f_i v]
            DIVOCCUR:=True
```

else

$i := i + 1$

end if

if *DIVOCCUR==False* **then**

$r := r + \mathrm{CTip}(z)\,\mathrm{Tip}(z)$

$z := z - \mathrm{CTip}(z)\,\mathrm{Tip}(z)$

end if

end for

end while

return r

Proposition 3.6. *Algorithm 3.1.1 terminates and is correct.*

Proof. For a proof see Green [Gre99]. □

***Example* 3.1.3.** *Take the noncommutative polynomial ring $\Bbbk\langle x, y, z\rangle$ over a field $\Bbbk$. Let $\mathcal{B}$ be the set of monomials and $>$ the length-lexicographic ordering with $x > y > z$. We divide $zxxyx$ by $f_1 = xy - x$ and $f_2 = xx - xz$. Note that $\mathrm{Tip}\,(f_1) = xy$ and $\mathrm{Tip}\,(f_2) = xx$. Beginning the algorithm, we see that $zxxyx = (zx)\,\mathrm{Tip}\,(f_1)\,x$. Thus $u_{1,1} = zx$ and $v_{1,1} = x$. We then replace $zxxyx$ by $zxxyx - zx\,(f_1)\,x = zxxx$. Now $\mathrm{Tip}\,(f_1)$ does not divide $zxxx$. We now consider $\mathrm{Tip}\,(f_2)$ and see it divides $zxxx$ and so we proceed. There are two ways to divide $zxxx$ by xx and for the algorithm to be precise we must choose one. Say we choose the "left most" division. Then $zxxx = zx\,(\mathrm{Tip}\,(f_2))$ and let $u_{2,1} = zx$ and $v_{2,1} = 1$ and replace $zxxx$ by $zxxz$. Once again we divide by $\mathrm{Tip}\,(f_2)$ and we see that $zxxz = z\,\mathrm{Tip}\,(f_2)\,z$ and so $u_{2,2} = z$ and $v_{2,2} = z$. We replace $zxxz$ by $zxzz$ and the algorithm terminates with $r = zxzz$. So we have*

$$zxxyx = (zx)\,f_1 x + zx f_2 + z f_2 z + zxzz.$$

If we change the order of f_1 and f_2 we get a different outcome:

$$zxxyx = z\,(f_2)\,yx + zxzyx.$$

Definition 3.1.11. *If $X = \{x_1, ..., x_n\} \subseteq A$ (as an ordered set) and $y \in A$ is divided by the set X, we denote the **remainder**, r, of the division of y by X as $y \Rightarrow_X r$.*

We note that if we divide by a set the outcome is not unique. It depends on the order in which we do the division. However, if we have a Gröbner basis, the outcome of the division algorithm is unique as the next proposition demonstrates.

Proposition 3.7. *Let $\mathcal{G}$ be a Gröbner basis for an ideal $I \in A$. Let $y \in A$ and consider $X = \{g_1, ..., g_n\} = \{g \in \mathcal{G} : \mathrm{Tip}\,(g) \leq \mathrm{Tip}\,(y)\}$. If $y \Rightarrow_X r$, then r is independent of the order of $g_1, ..., g_n$ and thus $r = N\,(y)$ is the **normal form** of y.*

Proof. For a proof of this proposition see Green [Gre99]. □

3.1.1 Computational Uses of Gröbner Bases

To study a quotient algebra $\Bbbk\Gamma/I$, a Gröbner basis provides a good method for working with the equivalence classes $f + I$. The information we gain if we have a Gröbner basis is summarized by the following proposition.

Proposition 3.8. *Let A be a $\Bbbk$-algebra with multiplicative basis $\mathcal{B}$ and admissible order $>$ on $\mathcal{B}$. Let I be an ideal in A with $\mathcal{G}$ a Gröbner basis for I. Let $f \Rightarrow_{\mathcal{G}} r$ with $r = N\,(f)$ the normal form of f. Then the following statements hold:*

1. *$f + I = g + I$ if and only if $N\,(f) = N\,(g)$.*

2. *$f + I = N\,(f) + I$.*

3. *The map $\sigma : A/I \to A$ with $\sigma\,(f + I) = N\,(f)$, is a vector space splitting to the canonical surjection $A \to A/I$.*

4. *σ is an $\Bbbk$-linear isomorphism between A/I and* $\operatorname{Span}(\operatorname{NonTip}(I))$*.*

5. *Identifying A/I with* $\operatorname{Span}(\operatorname{NonTip}(I))$*, then* $\operatorname{NonTip}(I)$ *is an $\Bbbk$-basis of A/I contained in $\mathcal{B}$.*

In general, Gröbner bases are used to perform calculations in an abstract finitely-presented algebra. So the construction of a Gröbner basis in our case may seem redundant. But the work of D. Anick, E. Green and others uses Gröbner bases to algorithmically construct projective resolutions of finitely presented $\Bbbk\Gamma/I$-modules. These results can be found in [Gre99], [AG87], [FGKK93], and a new method is found in [GSZ01].

3.1.2 Alternative Gröbner Basis Algorithm

The standard way of constructing a Gröbner basis for a generating set $\mathcal{G}$ is using the concept of an S-polynomial generalized to a noncommutative ring. There is a termination theorem which states that under certain conditions, if all S-polynomials reduce to zero, $\mathcal{G}$ is a Gröbner basis for $I = \langle \mathcal{G} \rangle$. The standard algorithm to compute a Gröbner basis is called the Buchberger algorithm (see [Gre99]). However, in the noncommutative setting our ring may not be Noetherian. So for a general $\Bbbk$-algebra we are not guaranteed a finite Gröbner basis.

In our case, the basic algebra B is a finite dimensional algebra. Fortunately, the following proposition gives a sufficient condition to have a finite Gröbner basis that relies only upon the finite dimensionality of the quotient.

Proposition 3.9. *Let A be a finitely generated $\Bbbk$-algebra with multiplicative basis and order $>$. Suppose that I is an ideal such that $\dim_{\Bbbk}(A/I)$ is finite. Then I has a finite Gröbner basis with respect to $>$.*

Proof. For a proof of this proposition see [Gre99]. $\square$

In the setting of this dissertation we are working with a finite dimensional basic algebra B. Therefore we are in a much better situation for computing Gröbner bases than for an arbitrary $\Bbbk$-algebra. Recall that the generators for a path algebra are the vertices (idempotents) and the arrows. The basic algebra that we work with is presented via a basis consisting of monomials in the generators. As we know the idempotents, we also know the PIMs. In addition, we know the action of the generators on this special basis. More specifically, we are given matrices for the action of the generators on the basis. Therefore, it would be redundant to use the Buchberger algorithm to construct our Gröbner basis. In our case we can do much better than the Buchberger algorithm.

Definition 3.1.12. *The set of **minimal tips** of I is:*

$$\mathrm{MinTip} = \{x \in \mathrm{Tip}(I) : \textit{the only } y \in \mathrm{Tip}(I) \textit{ dividing } x \textit{ is } x \textit{ itself}\}.$$

For $x \in \Bbbk\Gamma$, recall that $N_I(x)$ denotes the unique smallest support element of the coset $f + I$ as in lemma 3.2 and definition 3.1.8. This is the standard coset representative of $f + I$. The following definition makes sense due to the properties of $\mathrm{NonTip}(I)$ in lemma 3.2.

Definition 3.1.13. *An element $g \in I$ is **sharp** if it is of the form $x - N_I(x)$ for some $x \in \mathrm{Tip}(I)$. The set of minimal sharp elements is*

$$\mathrm{MinSharp} := \{x - N_I(x) : x \in \mathrm{MinTip}(I)\}.$$

The first thing we will need for our Gröbner basis algorithm is a way to compute $N_I(f)$ for $f \in \mathrm{NonTip}(B)$.

Algorithm 3.1.2. *Compute $N_I(f)$ Given a polynomial $f \in \mathrm{Tip}(I)$ in a basic $B = \Bbbk\Gamma/I$, we wish to compute the unique element of the coset $f + I$ with smallest support.*

Input: *$f \in \mathrm{MinTip}(I)$ for a basic algebra $B = \Bbbk\Gamma/I$ with a basis $\mathcal{B}$ consisting of monomials in the generators $Gen = \{v_1, ..., v_m, a_1, ..., a_n\}$ of B and matrices M_g for the action of the generators on $\mathcal{B}$.*

Output: $N_I(f)$

```
Pick x ∈ NonTip(I) such that there is a g ∈ Gen such that f = g · x.
Compute y := M_g · x.
N_I(f) := y, a linear combination of b_i ∈ B.
return N_I(f)
```

Lemma 3.10. *Algorithm 3.1.2 is correct and terminates.*

Proof. This follows from Lemma 3.2 and the fact that we are given a special basis.

□

We now present an alternative to the standard Buchberger algorithm for computing a Gröbner basis in our specific setting. In the algorithm below we let $\mathcal{B}$ be the basis for our path algebra $\Bbbk\Gamma$ and $\mathcal{B}_B$ the basis for our basic algebra B.

Algorithm 3.1.3. *Alternative Gröbner Basis Algorithm*

Input: *Basic Algebra $B = \Bbbk\Gamma/I$ with basis $\mathcal{B}_B$ ordered length lexicographically.*

Output: *Reduced Gröbner basis $\mathcal{G}$.*

```
NonTip(I) := B_B
MinTip(I) := ∅
MinSharp(I) := ∅
for all paths x in NonTip(I) (taken in the given ordering) do
  for all generators g of kΓ with τ(x) = o(g) (paths are compatible) do
    y := x · g (this is a basis element of kΓ)
    if y ∉ NonTip(I) then
      y is a tip
      if y not divisible by any element of MinTip(I) then
```

Add y to MinTip(I)

Compute $N_I(y)$, i.e. express y as linear combination of tips using Algorithm 3.1.2.

Add $y - N_I(y)$ to MinSharp(I).

end if

else

y is a NonTip(I) *as it is already in our basis*

end if

end for

end for

return *$\mathcal{G} := MinSharp(I)$, a Gröbner Basis I.*

Proposition 3.11. *The above algorithm 3.1.3 terminates and gives a Gröbner basis $\mathcal{G}$.*

Proof. As the basic algebra is finite dimensional and the path algebra is finitely generated, the algorithm clearly terminates after finitely many steps. Let $t \in \mathrm{Tip}(I)$. We know that t is divisible by a minimal tip. By construction we know there is a g in $\mathcal{G}$ such that $\mathrm{Tip}(g)$ divides t. So by Proposition 3.4, $\mathcal{G}$ is a Gröbner basis for I. □

3.2 Anick-Green and Minimal Resolutions

Let (Γ, I) be a special quiver with relations. Recall that the justification of this definition is that the quiver encodes a lot of information about the representation theory of the finite-dimensional $\Bbbk$-algebra $\Lambda = \Bbbk\Gamma/I$. In particular, the simple Λ-modules are just the vertex simples S_v, one for each vertex v of Γ which corresponds to a projective indecomposable module $e_v\Lambda$. Recall also that the first terms of the minimal projective resolution for S_v can be determined using quiver information. This partial resolution can be extended to a (not necessarily minimal) projective resolution, using our knowledge about the quiver and the relations. When computing a Gröbner basis, one often refers to computing overlaps or in the commutative case it is usually referred to as S-polynomials. We will generalize this notion to higher overlap sets and then use these sets as indices in our projective resolution. For the remainder of the chapter, we combine the ideas of E.Green [Gre94] and D. Green [Gre97].

3.2.1 Overlap sets

Definition 3.2.1. *Define Γ_0 to be the set of vertices in Γ, and Γ_1 to be the set of arrows. Define Γ_2 to be $\mathrm{MinTip}\,(I)$. For $n \geq 3$, define Γ_n to be the set of paths $\gamma \in \mathcal{B}$ such that*

1. *γ has a factorization $\gamma = \gamma_1\gamma_2$ with $\gamma_1 \in \Gamma_{n-1}$, $\gamma_2 \in \mathrm{NonTip}\,(I)$ and γ_2 has positive length.*
2. *For every factorization $\gamma = \gamma_1\gamma_2$ with $\gamma_1 \in \Gamma_{n-1}$, and for every factorization $\gamma_1 = \beta_1\beta_2$ with $\beta_1 \in \Gamma_{n-2}$, we have $\beta_2\gamma_2 \in \mathrm{Tip}\,(I)$.*
3. *No proper left factor of γ satisfies both* 1. *and* 2.

For implementation purposes later, we give an algorithm to compute the overlap sets. The idea is to not only keep track of a word γ in Γ_n, but also to keep track of the word that it came from in Γ_{n-1}. Then we check to see if there is any overlap between the part of the word γ that is the portion of the word in Γ_n but not in Γ_{n-1}. For example if we have a word $abbbc$ in Γ_4 that came from the word $abb \in \Gamma_3$, then we look for all words in Γ_2 that begin with bc. Then we only keep the largest possible overlaps; that is if a word $\phi \in \Gamma_n$ divides another word $\sigma \in \Gamma_n$, we throw out σ. We will refer to the previous word in γ from the previous level of Γ as $\mathrm{prev}\,(\gamma)$. We also assume that each word γ is a product of arrows $a_{i_1} \cdots a_{i_r}$. We denote the set of $\mathrm{prev}\,(\gamma_P) \in \Gamma_n$ as $\mathrm{prev}\,(\Gamma_n)$.

Algorithm 3.2.1. *Higher Overlaps*

Input: *The level of computation desired* n*, the basic algebra* $B = \Bbbk\Gamma/I$ *and basis* $\mathcal{B}_B$.

Output: *Higher overlaps* $\Gamma_1, \Gamma_2, ..., \Gamma_n$

Initialize: $\Gamma_1 := \{arrows\}$, $\Gamma_2 := \mathrm{MinTip}\,()$, $\mathrm{prev}\,(\Gamma_2) := \emptyset, \Gamma_m = \mathrm{prev}\,(\Gamma_m) := \emptyset$ *for* $3 \leq m \leq n$;

for $\gamma \in \Gamma_2$ ***do***

  $\mathrm{prev}\,(\gamma)$:=$a_{i_1} \cdots a_{i_{r-1}}$

  Add $\mathrm{prev}\,(\gamma)$ *to* $\mathrm{prev}\,(\Gamma)$

end for

for i *from* 3 *to* n ***do***

  for $\gamma \in \Gamma_{i-1}$ ***do***

    for $\phi \in \mathrm{MinTip}(I)$ ***do***

      if $\phi = \mathrm{prev}\,(\gamma) \cdot x$, *for some path* $x \in \mathcal{B}_B$ ***then***

        Add $\gamma\phi$ *to* Γ_i

        Add γ *to* $\mathrm{prev}\,(\Gamma_i)$

      end if

    end for

  end for

  Reduce Γ_i *(respectively* $\mathrm{prev}\,(\Gamma_i)$*), i.e. remove all words* $\gamma \in \Gamma_i$ *such that there is some* $\phi \in \Gamma_i$ *with* $\phi \neq \gamma$ *such that* $\phi \mid \gamma$.

end for

return $\Gamma_1, ..., \Gamma_n$

***Example* 3.2.1.** *Recall from Figure 3.1 in example 3.1.1 that the* Ext*-quiver for* $B \cong_{Morita} \mathbb{F}_2 S_4$ *has two vertices. It also has four arrows. Thus by definition we have that:*

$$\Gamma_1 = \{a, b, c, d\}$$

Then Γ_2 *is found by looking at the set* $\mathrm{MinTip}(I)$

$$\Gamma_2 = \{aa, cd, db, dd, cba, cbc, bac, bcb\}$$

Then we use algorithm 3.2.1 to compute Γ_3*:*

$$\Gamma_3 = \left\{ \begin{array}{c} aaa, cdb, cdd, dbac, dbcb, ddb, ddd, cbaa, cbac \\ cbcb, cbcd, bacd, bacba, bacbc, bcba, bcbc \end{array} \right\}$$

3.2.2 The Start of the Resolution

We first look at the construction of the Anick-Green resolution before we discuss the computation of a minimal projective resolution.

Let v be a vertex of Γ with corresponding idempotent e_v. As we saw in lemma 1.35, the vertex simple module S_v has projective cover $e_v\Lambda$, the epimorphism $e_v\Lambda \twoheadrightarrow S_v$ having kernel generated by the arrows a with origin v. That is, the sequence

$$\bigoplus_{a\in\Gamma_1} e_{\tau(a)}\Lambda \xrightarrow{\partial_1} \bigoplus_{v\in\Gamma_0} e_v\Lambda \xrightarrow{\varepsilon} \bigoplus_{v\in\Gamma_0} S_v \longrightarrow 0$$

is exact, where

$$\partial_1\left(e_{\tau(a)}\lambda\right) = e_{o(a)}a\lambda,$$

and ε is the sum of the projective covers of the simple module S_v. We describe the maps just by noting the images of the idempotents.

There is a $\Bbbk$-homomorphism $s_0 : \mathrm{Im}\,(\partial_1) \to \bigoplus_{a\in\Gamma_1} e_{\tau(a)}\Lambda$ which splits ∂_1. The set of positive-length paths $x \in \mathrm{NonTip}\,(I)$ is a basis for $\mathrm{Im}\,(\partial_1) = \mathrm{Ker}\,(\varepsilon)$. Each of these x may be uniquely expressed as $x = ay$, $a \in \Gamma_1$ and $y \in \mathrm{NonTip}\,(I)$. Let $s_0\,(x) = \tau\,(a)\,y$. Then $\partial_1\,(\tau\,(a)\,y) = ay = x$, so $\partial_1 s_0$ is the identity map on $\mathrm{Im}\,(\partial_1) = \mathrm{Ker}\,\varepsilon$.

Lemma 3.12. *Suppose that* ϕ *is a monic element of* $\mathrm{Ker}\,(\partial_1)$ *with* $\mathrm{Tip}\,\tau\,(a)\,y$*. Then* ay *has a (necessarily unique) factorization* $ay = \gamma z$*, with* $\gamma \in \Gamma_2$ *and* $z \in \mathrm{NonTip}\,(I)$*.*

Proof. Since a is an arrow and $I \subseteq J^2$, a must be a proper left factor of any such γ. Then any such z is a proper right factor of y, and so $z \in \operatorname{NonTip}(I)$. As no element of $\Gamma_2 = \operatorname{MinTip}(I)$ divides any other, it is enough to prove that any $ay \in \operatorname{Tip}(I)$. This is true as otherwise $\partial_1(\phi)$ cannot be zero. $\square$

We also know that each $\gamma \in \Gamma_2$ factorizes uniquely as ay, with $a \in \Gamma_1$ and $y \in \operatorname{NonTip}(I)$. Since $ay \in \operatorname{Tip}(I)$, it follows that $\tau(a)\,y - s_0\partial_1(\tau(a)\,y)$ is monic with $\operatorname{Tip}\tau(a)\,y$, So we can define a Λ-homomorphism

$$\bigoplus_{\gamma\in\Gamma_2} e_{\tau(\gamma)}\Lambda \xrightarrow{\partial_2} \bigoplus_{a\in\Gamma_1} e_{\tau(a)}\Lambda$$

by setting

$$\partial_2\left(e_{\tau(\gamma)}\right) := e_{\tau(a)}y - s_0\partial_1\left(e_{\tau(a)}y\right).$$

Then $\partial_1\partial_2 = 0$, since s_0 splits ∂_1. Moreover, the upshot of lemma 3.12 is that $\operatorname{Im}(\partial_2) = \operatorname{Ker}(\partial_1)$. To see this, we shall construct a $\Bbbk$-linear map $s_1 : \operatorname{Ker}(\partial_1) \to \bigoplus_{\gamma\in\Gamma_2} e_{\tau(\gamma)}\Lambda$ splitting ∂_2.

Algorithm 3.2.2. *Split* ∂_2

Input: $\phi \in \operatorname{Ker}(\partial_1)$

Output: $s_1(\phi)$ *such that* $\partial_2(s_1(\phi)) = \phi$

1: ***if*** $\phi = 0$ ***then***

2: $\quad s_1(\phi)$ *is* 0.

3: ***else***

4: $\quad \operatorname{Tip}(\phi)$ *is* $e_{\tau(a)}y$ *for unique* $a \in \Gamma_1$, $y \in \operatorname{NonTip}(I)$

5: $\quad c :=$ *coefficient of* $\tau(a)\,y$ *in* ϕ.

6: $\quad ay = \gamma z$ *for unique* $\gamma \in \Gamma_2$, $z \in \operatorname{NonTip}(I)$ *(Lemma 3.12).*

7: $\quad s_1(\phi)$ *is* $c \cdot e_{\tau(\gamma)}z + s_1\left(\phi - \partial_2\left(c \cdot e_{\tau(\gamma)}z\right)\right)$.

8: ***end if***

9: ***return*** $s_1(\phi)$

Lemma 3.13. *The Algorithm 3.2.2 terminates and is correct. The routine does define a* $\Bbbk$*-linear map* s_1 *such that* $\partial_2 s_1$ *is the identity on* $\mathrm{Ker}\,(\partial_1)$*.*

Proof. The algorithm terminates because at each stage $\mathrm{Tip}\,(\phi)$ decreases and we know that $\mathcal{B}$ is a well-ordered basis.

For $\Bbbk$-linearity we pick ϕ_1, ϕ_2 with $\mathrm{Tip}\,(\phi_1) \geq \mathrm{Tip}\,(\phi_2)$ such that $s_1\,(\phi_1 + \phi_2) \neq s_1\,(\phi_1) + s_1\,(\phi_2)$ and $\mathrm{Tip}\,(\phi_1)$ is as small as possible. If $\mathrm{Tip}\,(\phi_1 + \phi_2) = \mathrm{Tip}\,(\phi_1)$, then we can contradict the minimality of this counterexample. If $\mathrm{Tip}\,(\phi_1 + \phi_2) < \mathrm{Tip}\,(\phi_1)$, then $\mathrm{Tip}\,(\phi_1) = \mathrm{Tip}\,(\phi_2) = e_{\tau(a)}y$, occurring with coefficient c in ϕ_1, and $-c$ in ϕ_2. Factorize $ay = \gamma z$. Then get cancelation, and

$$s_1\,(\phi_1) + s_1\,(\phi_2) = s_1\left(\phi_1 - \partial_2\left(c \cdot e_{\tau(\gamma)}z\right)\right) + s_1\left(\phi_2 + \partial_2\left(c \cdot e_{\tau(\gamma)}z\right)\right).$$

By the minimality assumption, however, the right hand side is $s_1\,(\phi_1 + \phi_2)$. And then we can prove that s_1 splits ∂_2 by using a minimal counterexample argument. □

Corollary 3.14. $\mathrm{Im}\,(\partial_2) = \mathrm{Ker}\,(\partial_1)$*.*

Proof. We already have that $\mathrm{Im}\,(\partial_2) \subseteq \mathrm{Ker}\,(\partial_1)$ and so we show that $\mathrm{Ker}\,(\partial_1) \subseteq \mathrm{Im}\,(\partial_2)$. Let $x \in \mathrm{Ker}\,(\partial_1)$. Then by algorithm 3.2.2 we have that $s_1\,(x) \in \mathrm{Im}\,(\partial_2)$ and as $\partial_2\,(s_1\,(x)) = x$, we have that $\mathrm{Ker}\,(\partial_1) \subseteq \mathrm{Im}\,(\partial_2)$. □

***Example* 3.2.2.** *Continuing example 3.2.1 we have that*

$$\mathrm{MinSharp}\,(I) = \{aa, cd, db, dd + bc, cba + acb, cbc, bac + bc, bcb\}$$

Then

$$s_0\partial_1\left(e_{\tau(c)}ba\right) = s_0\,(cba) = s_0\,(acb) = e_{\tau(a)}cb,$$

so that

$$\partial_2\left(e_{\tau(cba)}\right) = e_{\tau(c)}ba + e_{\tau(a)}cb.$$

***Example* 3.2.3.** *Continuing the previous example 3.2.2 we have*

$$\partial_1\left(e_{\tau(c)}ba\right) = cba = acb,$$

and so $\phi := e_{\tau(c)}ba + e_{\tau(a)}cb$ *lies in* $\operatorname{Ker}(\partial_1)$*. So* $\operatorname{Tip}(\phi) = e_{\tau(c)}ba$*, and* cba *factors as* γz *with* $\gamma = acb$*,* $z = v_1$*. Then* $\partial_2\left(e_{\tau(cba)}\right) = e_{\tau(c)}ba + e_{\tau(a)}bc$*, so* $s_1(\phi) = e_{\tau(cba)} + s_1(0)$*. That is,*

$$s_1\left(e_{\tau(c)}ba + e_{\tau(a)}cb\right) = e_{\tau(cba)}.$$

Lemma 3.15. *Let* ϕ *be a nonzero element of* $\operatorname{Ker}(\partial_2)$*. Let* $\operatorname{Tip}(\phi)$ *be* $e_{\tau(\gamma)}x$*. Then* γx *has a left factor in* Γ_3*.*

Proof. We need to verify the first two conditions of definition of Γ_n for γx. They follow from the nature of the preferred basis, and from the fact that $\partial_2(\phi)$ does not have tip equal to $\operatorname{Tip}\left(\partial_2\left(e_{\tau(\gamma)}\right)\right)x$. $\square$

3.2.3 The Anick-Green resolution

Definition 3.2.2. *The terms of the* ***Anick-Green resolution*** *are the projective* Λ*-modules*

$$P_n = \bigoplus_{\gamma \in \Gamma_n} e_{\tau(\gamma)}\Lambda \text{ for } n \geq 0.$$

The preferred basis for P_n consists of the $e_{\tau(\gamma)}x$, where $\gamma \in \Gamma_n$, $x \in \operatorname{NonTip}(I)$ and $\tau(\gamma) = o(x)$.

Lemma 3.16. *Sending* $e_{\tau(\gamma)}x$ *to* γx *injects the preferred basis of* P_n *into* $\mathcal{B}$*. The admissible ordering on* $\mathcal{B}$ *therefore induces a well-ordering on the preferred basis of* P_n*.*

Proof. Follows from the definition of Γ_n. $\square$

Assume we have constructed $\partial_1, \partial_2, s_1$ as in the previous section 3.2.2. Let $n \geq 3$. Assume we have constructed differentials ∂_m for $m < n$, and that we have a $\Bbbk$-linear map s_{n-2} splitting ∂_{n-1}.

Each $\gamma \in \Gamma_n$ factors uniquely as $\gamma = \gamma_1\gamma_2$, with $\gamma_1 \in \Gamma_{n-1}$ and $\gamma_2 \in \mathrm{NonTip}\,(I)$. Define ∂_n by

$$\partial_n\left(\gamma\left(\tau\right)\right) := \tau\left(\gamma_1\right)\gamma_2 - s_{n-2}\partial_{n-1}\left(\tau\left(\gamma_1\right)\gamma_2\right).$$

with the splitting map as constructed in algorithm 3.2.2. Then as before we can show that

$$\mathrm{Im}\,\partial_n = \mathrm{Ker}\,\partial_{n-1}.$$

To summarize we state the following theorem.

Theorem 3.17. *The Anick-Green resolution, with differentials ∂_n and splitting s_{n-1} as constructed above is a Λ-projective resolution of $\bigoplus_{v\in\Gamma_0} S_v$.*

3.2.4 The Resolution of a Vertex Simple Module

In the previous section we have shown how to compute the projective resolution for $\bigoplus_{v\in\Gamma_0} S_v$. We shall now restrict our attention to focusing on just one S_v at a time. We also present the maps ∂_i that we defined in the previous section in an alternative way using more Gröbner basis theory. This is the method that we choose for implementation purposes in our program.

For each vertex v (respectively each idempotent e_v) we wish to define right Λ-module homomorphisms:

$$\partial_i(v) : \bigoplus_{\substack{p\in\Gamma_i\\ o(p)=v}} e_{\tau(p)}\Lambda \to \bigoplus_{\substack{q\in\Gamma_{i-1}\\ o(p)=v}} e_{\tau(q)}\Lambda, \text{ for } i = 1, 2, 3.$$

so that we get an exact sequence:

$$\bigoplus_{\substack{p\in\Gamma_3\\ o(p)=v}} e_{\tau(p)} \xrightarrow{\partial_3} \bigoplus_{\substack{p\in\Gamma_2\\ o(p)=v}} e_{\tau(p)} \xrightarrow{\partial_2} \bigoplus_{\substack{p\in\Gamma_1\\ o(p)=v}} e_{\tau(p)} \xrightarrow{\partial_1} \bigoplus_{\substack{p\in\Gamma_0\\ o(p)=v}} e_{\tau(p)} \xrightarrow{\varepsilon} S_v \to 0 \tag{3.2}$$

where S_v is the simple Λ-module associated to the vertex v, $e_{\tau(p)}$ is the idempotent corresponding to the vertex $\tau(p)$, and $\varepsilon(v) : e_v\Lambda \to S_v$ is the projective cover of S_v. To define the ∂_i we need only define $\partial_i(e_{\tau(p)})$ since if $\lambda \in \Lambda$ then $\partial_i\left(\left(e_{\tau(p)}\right)\lambda\right) = \partial_i\left(e_{\tau(p)}\right)\lambda$. Note that for all $o(p) = v$, we have simply that $\bigoplus_{p\in\Gamma_0} e_{\tau(p)}\Lambda = e_v\Lambda$.

∂_1**:** We have that the map ∂_1 is the same as above in the previous section. That is, if $a \in \Gamma_1$ such that $o(a) = v$, then a is an arrow from v to $\tau(a)$. Recall that $\partial_1(o(v)\cdot a) = e_v a$. As $a \in e_v\Lambda e_{\tau(a)}$ we know that if we define $\partial_1 v$ on the generators of $\bigoplus_{a\in\Gamma_1} e_{\tau(a)}\Lambda$ can be extended to a right Λ-module homomorphism.

∂_2**:** Next let $t \in \Gamma_2$ be an element of $\mathrm{MinSharp}(I)$ originating at vertex v. We know from the definition of Γ_2 that we can write $t = a^{'}b^{'}$ for $a^{'} \in \Gamma_1$ and $b^{'} \in \mathcal{B}$. We know that Γ_1 is just the set of arrows and so $a^{'}$ is just the first arrow in the path t. This is useful to know for computational purposes, as all we have to do is take off the first arrow from t. As $\mathcal{G} = \mathrm{MinSharp}(I)$ is the reduced Gröbner basis for I, there is a unique minimal sharp element $f_t = t + \sum_j \alpha_{j,t}p_{j,t}$ where $t > p_{j,t}$ for all $j, \alpha_{j,t} \in \Bbbk \setminus \{0\}$ and each $p_{j,t} \in \mathrm{NonTip}(I)$. For each $p_{j,t}$ there is an $a_{j,t} \in \Gamma_1$ and $q_{j,t} \in \mathcal{B}$ such that $p_{j,t} = a_{j,t}q_{j,t}$. Knowing this, we define the map ∂_2 as follows:

$$\partial_2(t) = e_{\tau\left(a^{'}\right)}b^{'} + \sum_j e_{\tau(a_{j,t})}\alpha_{j,t}q_{j,t} \in \bigoplus_{\substack{a\in\Gamma_1\\ o(a)=v}} e_{\tau(a)}\Lambda.$$

Since each path that occurs in f_t has origin v, we have that the arrows $a^{'}$ and $a_{j,t}$ have origin v and so $\partial_2(v)$ is well-defined on the generators of $\bigoplus_{t\in\Gamma_2} e_{\tau(t)}\Lambda$ where $o(t) = v$. Then we can extend $\partial_2(v)$ to a right Λ-module homomorphism.

∂_3**:** Let $p \in \Gamma_3$ be a word with origin v. We can write $p = tb = b^{'}t^{'}$ with $t \in \Gamma_2$. We know that t has origin vertex v. Now let f_t and $f_{t^{'}}$ be the minimal sharp elements of I with tips t and $t^{'}$ respectively. As we know that we have a Gröbner basis $\mathcal{G}$, by reduction theory (see [FFG93]) we can reduce $b^{'}f_{t'} - f_t b$ to 0 by elements of $\mathcal{G}$. We will now review some of the basic features of reduction that we will use.

Let $x \in I$ and $z = zw = vz$ for some vertices v and w, then we have a sequence of 4-tuples

$$\left(\gamma_1, c_1, f_1^{'}, d_1\right), \ldots, \left(\gamma_s, c_s, f)s^{'}, d_s\right)$$

where γ_j are nonzero elements of $\Bbbk$, $c_j, d_j \in \mathcal{B}$ and $f_j' \in \mathcal{G}$ satisfy the two following properties:

1. For $j = 0, ..., s-1$ the tip of $z - \left(\gamma_1 c_1 f_1' d_1 + \cdots + \gamma_j c_j f_j' d_j\right)$ is the tip of $c_{j+1} f_{j+1}' d_{j+1}$ with coefficient γ_{j+1} and
2. $z = \sum_{j=1}^{s} \gamma_j c_j f_j' d_j$.

We will say that in the above description z reduces to 0. We now return to $b' f_{t'} - f_t b$. As $b' f_{t'} - f_t b$ is in I, it reduces to 0 and we have that

$$b' f_{t'} - f_t b = \sum_{j,p} \alpha_{j,p} c_{j,p} f_{j,p} d_{j,p} \tag{3.3}$$

where $\alpha_{j,p} \in \Bbbk^*$, $c_{j,p}, d_{j,p} \in \mathcal{B}$ and $f_{j,p} \in \mathcal{G}$ with $\mathrm{Tip}\left(c_{j,p} f_{j,p} d_{j,p}\right) < p$. In general, (3.3) is not unique. However, a reduction must exist. And for our purposes the important thing is that it may be found algorithmically.

We shall do this by ordering the paths in $b' f_{t'} - f_t b$ and we will start with the largest path that is divisible by the tip of an element $m \in \mathrm{MinSharp}(I)$. Subtracting the appropriate multiple of m we continue the same process. We rewrite (3.3) as

$$b' f_t = f_t b + \sum +j, p \alpha_{j,p} c_{j,p} f_{j,p} d_{j,p}. \tag{3.4}$$

Lastly, we consider only the terms of right hand side of (3.4) where $c_{j,p}$ has length 0, i.e., $c_{j,p} = v$. We then write this sum as:

$$f_t b + \sum_{j',p} \alpha_{j',p} f_{j',p} d_{j',p}.$$

Every $f_{j',p} \in \mathrm{MinSharp}(I)$ has a tip $t_{j',p}$ which has origin v and is in Γ_2. We can finally define the map ∂_3 as follows:

$$\partial_3(p) = e_{\tau(t)} b + \sum_{j',p} e_{\tau\left(t_{j',p}\right)} \alpha_{j',p} d_{j',p} \in \bigoplus_{\substack{t \in \Gamma_2 \\ o(t)=v}} e_{\tau(t)} \Lambda.$$

Once again, as ∂_3 is well-defined on the generators, it can be extended to a right Λ-module homomorphism.

Theorem 3.18. *The sequence (3.2) on page 98 defined above is exact.*

Proof. For a proof see Green et al. [FGKK93, pages 1878-1879] □

The next lemma is of importance to us as we wish to compute minimal resolutions. We use it for both our linear algebra construction and for the Anick-Green methods.

Lemma 3.19. *The Anick-Green resolution for S_v,*

$$\cdots \to P_2 \to P_1 \to P_0 \to S_v \to 0$$

is minimal at P_1 and at P_0.

Proof. The radical of Λ is the ideal generated by the arrows. So $\mathrm{Im}\,(\partial_1) \subseteq \mathrm{Rad}\,(P_0)$ from the construction of ∂_1 and $\mathrm{Im}\,(\partial_2) \subseteq \mathrm{Rad}\,(P_1)$ since $I \subseteq J^2$. □

This construction will give a projective resolution for any given degree n, however, the Anick-Green resolution is not necessarily minimal at P_2 as the next example shows. However, as we shall see in the next section, we are able to use lemma 3.19 along with the technique of one-point extension to compute a minimal resolution.

***Example* 3.2.4.** *Consider $B \cong_{Morita} \mathbb{F}_2 S_4$ with* Ext-*quiver as in example 3.1.1. Let S_{v_1} be the simple module corresponding to the idempotent e_{v_1}. Then if we compute the projective resolution of S_{v_1} we get the following:*

$$e_{\tau(aa)}\Lambda \oplus e_{\tau(cd)}\Lambda \oplus e_{\tau(cba)}\Lambda \oplus e_{\tau(cbc)}\Lambda \xrightarrow{\partial_2} e_{\tau(a)}\Lambda \oplus e_{\tau(c)}\Lambda \xrightarrow{\partial_1} e_{v_1}\Lambda \xrightarrow{\varepsilon} S_{v_1} \longrightarrow 0$$

However, we shall see in our example at the end of the chapter that $e_{\tau(cbc)}$ is a redundant generator and is therefore not minimal at (P_2, ∂_2).

3.2.5 Resolution for Finitely Presented modules

The Anick-Green resolution has two limitations: it is not minimal, and it only exists for vertex simple modules. Of these limitations, not being minimal is more serious in its effect in cohomological computations. We give an example of how quickly the Anick-Green resolution can grow:

***Example* 3.2.5.** *Consider the projective resolution of the simple B-module S_{v_1} as in example 3.2.4. Continuing using the Anick-Green resolution, the number of PIMs in each of the projective modules P_n in the resolution is:*

$$\{1, 2, 4, 7, 13, 26, 52, 103, 205, 410, 820, 1639, 3277, 6554\}$$

However, the number of PIMs for the minimal resolution is:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$$

We will first explain how to use the ideas of Anick and Green to compute projective resolutions for arbitrary modules that are not necessarily vertex simple. Then we will show how this allows us to compute minimal projective resolutions.

The goal is to construct a resolution for an arbitrary finitely presented Λ-module M. We shall construct a quiver Γ^* with one more vertex than Γ, in such a way that M is the Heller module $\Omega(M)$ of the new vertex simple. Since the first terms of the

Anick-Green resolution are minimal, we can iteratively compute a minimal resolution for M.

Definition 3.2.3. *Let M be a finitely generated Λ-module. We call*

$$\bigoplus_{j\in\mathcal{J}} v_j\Lambda \xrightarrow{F} \bigoplus_{i\in\mathcal{I}} v_i\Lambda \xrightarrow{\Phi} M \to 0 \tag{3.5}$$

a ***presentation*** *of M where $M = \bigoplus_{i\in\mathcal{I}} v_i\Lambda/\operatorname{Im} F$.*

Using this approach, the information that is important to us is the presentation of the module, i.e., the crucial information is F. We first introduce the notion of a one-point extension of the algebra Λ.

Definition 3.2.4. *Let M be a finitely presented Λ-module, as in (3.5). Define the* ***one-point extension of*** *Λ, denoted Λ^*, to be the $\Bbbk$-algebra of matrices $\begin{pmatrix} \lambda & m \\ 0 & f \end{pmatrix}$, with $\lambda \in \Bbbk$, $f \in \Lambda$ and $m \in M$, with usual matrix addition and multiplication, e.g.*

$$\begin{pmatrix} \lambda_1 & m_1 \\ 0 & f_1 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 & m_2 \\ 0 & f_2 \end{pmatrix} = \begin{pmatrix} \lambda_1\lambda_2 & \lambda_1 m_2 + m_1 f_2 \\ 0 & f_1 f_2 \end{pmatrix}.$$

Now we state some basic facts and properties of our new algebra.

Lemma 3.20. *Denote by v^* the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \Lambda^*$. Let S_{v^*} be the degree 1 Λ^*-module on which $\begin{pmatrix} \lambda & m \\ 0 & f \end{pmatrix}$ acts as multiplication by λ. Then*

1. *The map $f \mapsto \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix}$ identifies Λ with a subring of Λ^*, and so all Λ^*-modules are also Λ-modules.*

2. *The idempotent v^* is primitive. The Λ^*-module S_{v^*} is well-defined.*

3. *The projective cover of S_{v^*} is $v^*\Lambda^*$.*

4. *The matrices $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ in Λ^* form a Λ^*-submodule, the Heller module ΩS_{v^*}.*

5. As a Λ-module, $\Omega S_{v^*} \cong M$.

Proof. See D. Green [Gre97, page 43] $\square$

We would now like to see how to concretely construct the one point extension of the path algebra $\Bbbk\Gamma$. We will extend the quiver Γ to a quiver Γ^* by adding one vertex to Γ such that Γ^* only has arrows coming out of it and none going into it. Then we define an ideal I^* such that we get Λ^* as the quotient of $\Bbbk\Gamma^*$.

Define Γ^* to be the quiver obtained from Γ by adding one new vertex v^*, and by adding one arrow $v^* \xrightarrow{a_i^*} v_i$ for each $i \in \mathcal{I}$.

The path algebra $\Bbbk\Gamma^*$ contains $\Bbbk\Gamma$ as a subalgebra. Define I^* to be the ideal in $\Bbbk\Gamma^*$ generated by I, together with $\sum_{i\in\mathcal{I}} a_i^* f_{i,j}$ for each $j \in \mathcal{J}$. Here, the $f_{i,j} \in \Bbbk\Gamma$ with support in $\mathrm{NonTip}\,(I)$ are uniquely determined by

$$F\,(v_j) = \sum_i v_i f_{i,j} v_j \quad \text{and} \quad f_{ij} = v_i f_{ij} v_j.$$

Denote by π the projection $\Bbbk\Gamma \longrightarrow \Bbbk\Gamma/I \cong \Lambda$. We need to know the induced map π^* that we obtain as we go from $\Bbbk\Gamma$ to $\Bbbk\Gamma^*$. This is given in the following proposition.

Proposition 3.21. *There is a unique $\Bbbk$-algebra homomorphism $\pi^* : \Bbbk\Gamma^* \to \Lambda^*$ which sends v^* to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, a_i^* to $\begin{pmatrix} 0 & \Phi\,(v_i) \\ 0 & 0 \end{pmatrix}$ and $f \in \Bbbk\Gamma$ to $\begin{pmatrix} 0 & 0 \\ 0 & \pi\,(f) \end{pmatrix}$. This homomorphism is surjective with kernel I^*. That is,*

$$\Bbbk\Gamma^*/I^* \cong_{\pi^*} \Lambda^*.$$

Suppose that in the presentation (3.5), the number $|\mathcal{I}|$ of generators of M is the smallest possible. That is, suppose that $\mathrm{Ker}\,(\Phi) \subseteq \mathrm{Rad}\,(\oplus_i v_i \Lambda)$. Then the pair (Γ^*, I^*) is a special quiver with relations.

Proof. For a proof see [Gre97]. $\square$

***Example* 3.2.6.** *Let Γ be the quiver*

$$\begin{tikzcd} e_1 \arrow[r, shift left, "a"] & e_2 \arrow[l, shift left, "b"] \end{tikzcd}$$

Let $\Bbbk = \mathbb{F}_3$. Take $I = \langle aba, bab \rangle$, so that $\Lambda = \Bbbk\Gamma/I$ is the basic algebra for the group algebra $\Bbbk S_3$. Take S to be the trivial module with presentation

$$e_2\Lambda \xrightarrow{F} e_1\Lambda \xrightarrow{\Phi} \Bbbk \longrightarrow 0$$

where $F(e_2\lambda) = a \cdot \lambda$. Then the quiver Γ^* is

$$\begin{tikzcd} \overset{e_1}{\bullet} \arrow[r, shift left, "a"] & \overset{e_2}{\bullet} \arrow[l, shift left, "b"] \\ e_1^* \arrow[u, "a^*"'] & \end{tikzcd}$$

and the ideal I^* is generated by aba, bab, and a^*a.

Lemma 3.22. *1. The subalgebra Λ is also a right ideal in Λ^*.*

2. Any path in Γ^* with positive length has terminus vertex in Γ.

3. For any vertex $v \in \Gamma$, the projective Λ^*-module $v\Lambda^*$ is equal to $v\Lambda$ as a Λ-module.

Proof. For the last part, observe that $v\Lambda^* v^* = 0$. $\square$

The last thing we need to introduce is an admissible ordering on the set $\mathcal{B}^*$. We construct the ordering as an extension of the admissible ordering $>$ on $\mathcal{B}$.

There are three types of paths in Γ^* :

1. paths $\gamma \in \mathcal{B}$;

2. the vertex path v^*; and

3. paths $a_i^*\gamma'$ for $i \in \mathcal{I}$, $\gamma' \in \mathcal{B}$.

Pick an ordering on the finite set $\mathcal{I}$. Then define the ordering $\leq^*$ on $\mathcal{B}^*$ as follows:

1. $\leq^*$ extends $\leq$ on $\mathcal{B}$;

2. $\gamma <^* v^* <^* a_i^* \gamma'$;

3. $a_{1_i}^* \gamma' \leq^* a_{i_2}^* \gamma_2'$ if $i_1 < i_2$, or if $i_1 = i_2$ and $\gamma_1' \leq \gamma_2'$.

Lemma 3.23. *The ordering $\leq^*$ on $\mathcal{B}^*$ is admissible, and extends the ordering $\leq$ on $\mathcal{B}$. We have*

$$\mathrm{MinTip}\,(I) = \mathrm{MinTip}\,(I^*) \cap \mathcal{B},$$

and

$$\mathrm{MinSharp}\,(I) \subseteq \mathrm{MinSharp}\,(I^*)\,.$$

Proof. For a proof see D. Green [Gre97]. □

***Example* 3.2.7.** *Let Γ be the quiver given in Figure 3.1 in example 3.1.1 and take the length-lexicographic ordering on $\mathcal{B} = \{v_{1,2}\,, a, b, c, d, aa, ...\}$ as before*

$$v_1 < v_2 < a < b < c < d < aa < ac < ba < \cdots$$

Define a new ordering $\preceq$ on $\mathcal{B}^* = \mathcal{B} \cup \{(v^*, a^*, a^*a, a^*c, a^*aa, ...)^r\,, \text{for } r \in \mathbb{Z}^+\}$ as

$$v^* \prec a^* \prec a^*a \prec a^*c \prec a^*aa \prec \cdots \prec v_1 \prec v_2 \prec a \prec b \prec c \prec d \prec aa \prec \cdots$$

This ordering is admissible, however, it is not a length-lexicographic ordering.

Now we can finally obtain a projective resolution of any finitely presented Λ-module.

Theorem 3.24. *Let M be a finitely-presented Λ-module, presented with a smallest set of generators in the exact sequence*

$$\bigoplus_{j \in \mathcal{J}} v_j \Lambda \xrightarrow{F} \bigoplus_{i \in \mathcal{I}} v_i \Lambda \xrightarrow{\Phi} X.$$

As above, construct the special quiver with relations (Γ^*, I^*) *and algebra* Λ^* *with* $\Bbbk\Gamma^*/I^* \cong \Lambda^*$. *Construct the Anick-Green resolution for the new vertex simple* S_{v^*} :

$$\cdots \longrightarrow P_n^{v^*} \xrightarrow{\partial_n} P_{n-1}^{v^*} \longrightarrow \cdots \longrightarrow P_1^{v^*} \xrightarrow{\partial_1} P_0^{v^*} \xrightarrow{\varepsilon} S_{v^*} \longrightarrow 0.$$

Then $P_0^{v^*}$ *is* $v^*\Lambda^*$, $P_1^{v^*}$ *is* $\bigoplus_{i\in\mathcal{I}} v_i\Lambda$, *and* $\operatorname{Im}(\partial_1)$ *is* M. *Therefore*

$$\cdots \longrightarrow P_n^{v^*} \xrightarrow{\partial_n} P_{n-1}^{v^*} \longrightarrow \cdots \longrightarrow P_2^{v^*} \xrightarrow{d2} \bigoplus_{i\in\mathcal{I}} v_i\Lambda \xrightarrow{\Phi} M \longrightarrow 0$$

is a Λ*-projective resolution of* M. *Moreover, each* $P_r^{v^*}$ *is a direct sum of modules of the form* $v\Lambda$ *for vertices* $v \in \Gamma$.

3.2.6 Minimal Projective Resolutions

Now that we know how to construct a resolution for any module, we would like to know how to get rid of redundant generators in the resolution using the Gröbner basis approach. We will then combine this with the Anick-Green resolution and therefore have a way of constructing a minimal resolution.

The following proposition gives us a way to get rid of redundant generators.

Proposition 3.25. *Let* M *be a finitely presented* Λ*-module in the exact sequence*

$$\bigoplus_{j\in\mathcal{J}} v_j\Lambda \xrightarrow{F} \bigoplus_{i\in\mathcal{I}} v_i\Lambda \xrightarrow{\Phi} M.$$

Also denote by F *the matrix* (f_{ij}), *where* $F(v_j) = \sum_i v_i f_{ij} v_j$.

Suppose that this presentation involves redundant generators. In other words, suppose that $\operatorname{Im}(F)$ *is not contained in* $\operatorname{Rad}\left(\bigoplus_{i\in\mathcal{I}} v_i\Lambda\right)$. *Then there exists* $i_0 \in \mathcal{I}$, $j_0 \in \mathcal{J}$ *such that* $f_{i_0j_0}$ *is invertible, i.e.* $f_{i_0j_0} = \lambda e_{v_{i_0}} + x$ *for some* $\lambda \in \Bbbk^\times$ *and* $x \in \operatorname{Rad}(\Lambda)$. *Let* $g \in \Lambda$ *be such that* $f_{i_0j_0}g = gf_{i_0j_0} = e_{v_{i_0}}$.

Set $\mathcal{I}' := \mathcal{I} \setminus i_0$, $\mathcal{J}' := \mathcal{J} \setminus j_0$. *Define the matrix* $F' = \left(f'_{ij}\right)$ *for* $i \in \mathcal{I}'$ *and* $j \in \mathcal{J}'$ *by*

$$f'_{ij} = f_{ij} - f_{ij_0} g f_{i_0j}.$$

Then

$$\bigoplus_{j\in\mathcal{J}'} v_j\Lambda \xrightarrow{F'} \bigoplus_{i\in\mathcal{I}'} v_i\Lambda \xrightarrow{\Phi|_{\mathcal{I}'}} X$$

is exact, and $\operatorname{Im}(\Phi|_{\mathcal{I}'}) = M$*.*

Proof. For a proof see D. Green [Gre97, pages 46-47]. □

We write down an algorithm to reduce the matrix of a given resolution that is minimal. We first need to check if a matrix has an invertible entry.

Algorithm 3.2.3. *Matrix with Invertible Entries*
We would like to determine if an $m \times n$ *matrix* F *has a non-nilpotent entry. An entry is not nilpotent if it is of the form* $\lambda + k$ *for* λ *nilpotent and* $k \in \Bbbk$*.*

Input: *An* $m \times n$ *matrix* $F = [f_{i,j}]$

Output: *True if all entries are nilpotent or the set* $\{i, j\}$ *(the position of the first non-nilpotent entry) if an entry is not nilpotent.*

for *i from 1 to m* ***do***
    for *j from 1 to n* ***do***
        if $f_{i,j} = \lambda + k$ *for* λ *nilpotent and* $k \in \Bbbk$ ***then***
            return $\{i, j\}$
        end if
    end for
end for
return *True*

We now describe how to completely reduce a matrix F for a minimal presentation of a module M.

Algorithm 3.2.4. *Reduce Matrix*

Input: $F = [f_{i,j}]$*, an* $m \times n$ *matrix which gives the presentation for* M *above.*

Output: *A matrix* F' *which gives a minimal presentation of* M*.*

while *F has a non-nilpotent entry using Algorithm 3.2.3* ***do***

  $\{i_0, j_0\} := IsNilpotent\,(F)$

  $m := m - 1$;

  $n := n - 1$;

  $g := f_{i_0,j_0}^{-1}$ *using Lemma 1.14*

  for r *from* 1 *to* m ***do***

    if $r < i_0$ ***then***

      $k := r$;

    else if $r \geq i_0$ ***then***

      $k := r + 1$

    end if

    for s *from* 1 *to* n ***do***

      if $l < j_0$ ***then***

        $l := s$;

      else if $s \geq j_0$ ***then***

        $l := s + 1$

      end if

      $f'_{r,s} := f_{k,l} - f_{k,j_0} \cdot g \cdot f_{i_0,l}$.

    end for

  end for

  $F := \left[f'_{r,s}\right]$

end while

return F

Therefore we can turn any presentation of M into one with a minimal sized generating set. Therefore we now have an algorithmic way of taking any finitely presented module M and constructing a minimal projective resolution of it.

Let M be a Λ-module, finitely presented as

$$Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0.$$

The following algorithm constructs as many steps as desired in the minimal resolution $(P_\bullet, \varepsilon)$ of M.

Algorithm 3.2.5. *Anick-Green Minimal Resolution*

Input: *A finite minimal presentation of a Λ-module M as in (3.5) and degree n of computation desired.*

Output: *A minimal projective resolution $(P_\bullet, \varepsilon)$ up to degree n.*

1: *Compute first 2 steps of Anick-Green resolution using Theorem 3.24.*

$$Q_2 \longrightarrow S_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

2: *Obtain presentation of $\Omega(M)$*

$$Q_2 \longrightarrow S_1 \longrightarrow P_0.$$

3: *Minimize presentation of $\Omega(M)$ as in Proposition 3.25 using Algorithm 3.2.4 to get minimal presentation of $\Omega(M)$*

$$R_2 \longrightarrow P_1 \longrightarrow P_0.$$

4: *$P_1 \longrightarrow P_0$ is beginning of minimal projective resolution.*

5: *Repeat until desired level of resolution equals n.*

Example: The best way to get a feel for this algorithm is to work through an example. We shall continue our example of the basic algebra B for S_4 in characteristic 2 as in example 3.1.1 to demonstrate algorithm 3.2.5 in action.

Let $\Bbbk = \mathbb{F}_2$ and $G = S_4$. Let B be the basic algebra for $\mathbb{F}_2 S_4$. As we saw in example 3.1.1 we have 2 vertices and 4 arrows. We also have that $\Bbbk\Gamma/I = \Lambda$ where

$$\operatorname{MinSharp}(I) = \{aa, cd, db, dd + bc, cba + acb, cbc, bac + bc, bcb\} \tag{3.6}$$

We begin the minimal resolution for the vertex simple module S_{v_1} which has presentation

$$e_{\tau(a)}\Lambda \oplus e_{\tau(c)}\Lambda \xrightarrow{F} e_{v_1}\Lambda \xrightarrow{\Phi} S_{v_1} \longrightarrow 0, \tag{3.7}$$

where $F\left(\tau\left(c\right)\right) = c$, $F\left(\tau\left(e\right)\right) = e$ and $\Phi\left(c\right) = \Phi\left(e\right) = 0$. It is evident that there are no redundant summands to discard in $Q_0 = e_{v_1}\Lambda$.

We now form the one-point extension associated to the presentation of S_{v_1}. As $P_0 = e_{v_1}\Lambda$ has just one summand, the quiver Γ^* is

$$\begin{tikzcd} v_1 \arrow[loop left, "a"] \arrow[r, shift left, "c"] & v_2 \arrow[l, shift left, "b"] \arrow[loop right, "d"] \\ v_* \arrow[u, "a^*"] & \end{tikzcd} \tag{3.8}$$

The relations ideal I^* is then generated by $\mathrm{MinSharp}\left(I\right)$ together with the paths a^*a and a^*c . These 10 generators for I^* satisfy the definition of a small Gröbner basis. Thus the map $e_{\tau(a)}\Lambda \oplus e_{\tau(c)}\Lambda \xrightarrow{F} e_{v_1}\Lambda$ is the map $S_1 \rightarrow P_0$ obtained from the Anick-Green resolution.

We have the set $\Gamma_2^* = \{a^*a, a^*c\}$ and we wish to compute the higher overlaps. We obtain

$$\Gamma_3^* = \{a^*aa, a^*cd, a^*cba, a^*cbc\} \tag{3.9}$$

The term P_3^* in the Anick-Green resolution has one summand Λ for each element of Γ_3^*. The generator corresponding to a^*aa is denoted $\tau\left(a^*aa\right)$ (for simplification of notation at times we leave off the idempotent e and denote e_v as v). We now recall how to construct $\partial_3\left(\tau\left(a^*aa\right)\right)$. We consider the image $F\left(a^*aa\right) = \tau(a^*)aa$. Then we write $\tau(a^*)aa = tb = b^{'}t^{'}$ for t and b as in section 3.2.4. We have

$$tb \quad = \quad \tau(a^*)a \cdot a$$

and

$$b^{'}t^{'} \quad = \quad \tau(a^*) \cdot aa.$$

Next we compute $b^{'}f_{t'} - f_t b$ as before:

$$b^{'}f_{t'} - f_t b \quad = \quad \tau(a^*) \cdot aa - \tau(a^*)a \cdot a = 0$$

Thus $\partial_3 \left(\tau \left(a^* aa\right)\right) = \tau \left(a^* a\right) a$. Similarly, $\partial_3 \left(\tau \left(a^* cd\right)\right) = \tau \left(a^* c\right) d$.

To compute $\partial_3 \left(\tau \left(a^* cba\right)\right)$ we do the same as above for $F \left(a^* cba\right) = \tau \left(a^*\right) cba$.

$$tb \quad = \quad \tau(a^*)c \cdot ba$$

and

$$b^{'}t^{'} \quad = \quad \tau(a^*) \cdot cba.$$

Next we compute $b^{'}f_{t'} - f_t b$ as before. Recall that f_t and $f_{t'}$ are the minimal sharp elements of I^* with tips t and $t^{'}$ respectively:

$$b^{'}f_{t'} - f_t b \quad = \quad \tau(a^*) \cdot (cba + abc) - \tau(a^*) \cdot cba = \tau(a^*)abc$$

Thus we have that

$$\tau \left(a^* cba\right) \mapsto \tau \left(a^* c\right) ba + \tau \left(a^* a\right) bc$$

Lastly, we note that $\tau \left(a^* cbc\right) \mapsto \tau \left(a^* c\right) bc$.

So $Q_2 = P_3^{v^*} = e_{v_1}\Lambda \oplus e_{v_2}\Lambda \oplus e_{v_2}\Lambda \oplus e_{v_2}\Lambda$ as $\tau \left(a^* a\right) = e_{v_2}$ and $\tau \left(a^* c\right) = e_{v_2}$. The presentation we get of $\Omega S_{v_1} = \mathrm{Im} \left(F\right)$ is

$$Q_2 \longrightarrow e_{\tau(a^* a)}\Lambda \oplus e_{\tau(a^* c)}\Lambda. \tag{3.10}$$

The map $Q_2 \longrightarrow e_{\tau(a^* a)} \oplus e_{\tau(a^* c)} = e_{v_1}\Lambda \oplus e_{v_2}\Lambda$ has the matrix

$$\begin{bmatrix} a & 0 & cb & 0 \\ 0 & d & ba & bc \end{bmatrix} \tag{3.11}$$

As all elements in this matrix are nilpotent (i.e. in the radical), there are no redundant summands in $S_1 = e_{\tau(a^* a)}\Lambda \oplus e_{\tau(a^* c)}\Lambda$. Hence P_1 is $e_{\tau(a^* a)}\Lambda \oplus e_{\tau(a^* c)}\Lambda = e_{v_1}\Lambda \oplus e_{v_2}\Lambda$.

Now we forget about the one point extension (3.8) and construct a new one using the presentation (3.10) of ΩS_{v_1}. The new quiver Γ^* is

$$\begin{tikzcd} v_1 \arrow[loop left, "a"] \arrow[r, shift left, "c"] & v_2 \arrow[l, shift left, "b"] \arrow[loop right, "d"] \\ v_* \arrow[u, shift right, "a^*a"'] \arrow[u, shift left, "a^*c"] & \end{tikzcd} \tag{3.12}$$

where for notational ease we let $a^*a = A$ and $a^*c = C$. The relations ideal I^* is generated by I together with the 4 elements Aa, Cd, $Cba + Acb$, and Cbc. We have ordered the paths using the ordering $\leq^*$ with $A <^* C$.

We then know that MinSharp$\,(I)$ and these 4 generators generate I^*. In addition they form a reduced Gröbner basis. Thus the Gröbner basis is

$$\text{MinSharp}\,(I^*) = \text{MinSharp}\,(I) \cup \{Cbc, Cba + Acb, Cd, Aa\} \tag{3.13}$$

Thus

$$\Gamma_2^* = \{Cbc, Cba, Cd, Aa\} \tag{3.14}$$

$$\Gamma_3^* = \{Cbcd, Cbcb, Cbaa, Cbac, Cdb, Cdd, Aaa\} \tag{3.15}$$

We construct a presentation of $\Omega^2 S_{v_1}$ using the Anick-Green resolution. The term S_2 is

$$S_2 = e_{\tau(Cbc)}\Lambda \oplus e_{\tau(Cba)} \oplus e_{\tau(Cd)} \oplus e_{\tau(Aa)} = e_{v_2}\Lambda \oplus e_{v_1}\Lambda \oplus e_{v_2}\Lambda \oplus e_{v_1}\Lambda \tag{3.16}$$

The map to $P_1 = e_{\tau(A)}\Lambda \oplus e_{\tau(C)}\Lambda = e_{v_1}\Lambda \oplus e_{v_2}\Lambda$ comes by replacing Cbc, $Cba + Acb$, Cd, and Aa by their values in P_1. Now we want $Q_3 \to S_2$, where Q_3 has a summand for each of the elements in Γ_3^*

$$\begin{aligned} Q_3 &= \tau(Cbcd)\Lambda \oplus \tau(Cbcb)\Lambda \oplus \tau(Cbaa)\Lambda \oplus \tau(Cbac)\Lambda \oplus \tau(Cdb)\Lambda \\ &\quad \oplus \tau(Cdd)\Lambda \oplus \tau(Aaa) \\ &= e_{v_2}\Lambda \oplus e_{v_1}\Lambda \oplus e_{v_1}\Lambda \oplus e_{v_2}\Lambda \oplus e_{v_1}\Lambda \oplus e_{v_2}\Lambda \oplus e_{v_1}\Lambda \end{aligned}$$

We would like to compute the images of the idempotents in Q_3. We wish to compute the image of $\tau\left(Cbcd\right)$. The value of $\tau\left(Cbcd\right)$ in P_1 is $\tau\left(C\right)bcd$ and we have that

$$b^{'}f_{t'} - f_t b = \tau(C)bcd - \tau(C)bcd = 0.$$

So $\tau\left(Cbcd\right) \mapsto \tau\left(Cbc\right)d$. Similarly, as $b^{'}f_{t'} - f_t b = 0$ for the following we have:

$$\begin{aligned} \tau\left(Cbcb\right) &\mapsto \tau\left(Cbc\right)b \\ \tau\left(Cdb\right) &\mapsto \tau\left(Cd\right)b \\ \tau\left(Aaa\right) &\mapsto \tau\left(Aa\right)a \end{aligned}$$

To compute the image of $\tau(Cbaa)$ we look at the value of $\tau(Cbaa)$ in P_1 which is $\tau(C)baa$. We have that

$$\begin{aligned} tb &= \tau\left(C\right)ba \cdot a \\ b't' &= \tau\left(C\right)b \cdot aa \\ b'f_{t'} - f_t b &= \tau\left(C\right)baa - \left(\tau\left(C\right)baa + \tau\left(A\right)cba\right) \\ &= \tau\left(A\right)cba = \tau\left(A\right)acb \end{aligned}$$

Now we need to write this as an algebra sum of I^*. We do this by division. We see that $\tau(A)acb = \tau(A)a \cdot cb$. And thus we have that

$$\tau(C)baa \mapsto \tau(Cba)a + \tau(Aa)cb$$

Similarly we have that

$$\begin{aligned} \tau(Cbac) &\mapsto \tau(Cba) \cdot c + \tau(Cbc) \cdot 1 \\ \tau(Cdd) &\mapsto \tau(Cd) \cdot d + \tau(Cbc) \cdot 1 \end{aligned}$$

Thus the matrix for the map is:

$$f : \begin{bmatrix} d & b & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & a & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & d & 0 \\ 0 & 0 & cb & 0 & 0 & 0 & a \end{bmatrix} \tag{3.17}$$

Thus the resolution we have so far is not minimal at S_2, since $f_{1,4}$ and $f_{1,6} = 1$ are invertible. Thus the generator $\tau(Cbc)$ is superfluous and not needed. We replace the above matrix with the 3×6 matrix (using Proposition 3.25) with entries

$$f'_{ij} = f_{i'j'} - f_{i'4}f_{1j'} \text{ , where } i' = i+1 \text{ and } j' = \left\{ \begin{array}{c} j,\, j < 4 \\ j+1,\, j \geq 4 \end{array} \right\}. \tag{3.18}$$

For example,

$$\begin{aligned} f'_{1,1} &= f_{2,1} - f_{2,4}f_{1,1} = 0 - c \cdot d = cd = 0 \\ f'_{1,2} &= f_{2,1} - f_{2,4}f_{1,2} = 0 - c\dot{b} = cb \\ f'_{3,6} &= f_{4,7} - f_{4,4}f_{1,7} = a - 0 \cdot 0 = a - 0 = a \end{aligned}$$

The resulting matrix is

$$\begin{bmatrix} 0 & cb & a & 0 & c & 0 \\ 0 & 0 & 0 & b & d & 0 \\ 0 & 0 & cb & 0 & 0 & a \end{bmatrix} \tag{3.19}$$

All elements of this matrix are nilpotent and therefore we can conclude that P_2 is $e_{\tau(Cba)}\Lambda \oplus e_{tau(Cd)}\Lambda \oplus e_{\tau(Aa)}\Lambda$ which is just $e_{v_1}\Lambda \oplus e_{v_2}\Lambda \oplus e_{v_1}\Lambda$. The zero column corresponds to a summand whose image in P_2 is zero, and can therefore be deleted. Therefore we have the map from $P_2 \xrightarrow{\partial_2} P_1$ given by the matrix

$$\begin{bmatrix} ba & d & 0 \\ cb & 0 & a \end{bmatrix} \tag{3.20}$$

Continuing this process we compute

$$\begin{aligned} P_3 &= e_{\tau(Cdd)}\Lambda \oplus e_{\tau(Cdd)}\Lambda \oplus e_{\tau(Cdb)}\Lambda \oplus e_{\tau(Cbaa)}\Lambda \oplus e_{\tau(Aaa)}\Lambda \\ &= e_{v_2}\Lambda \oplus e_{v_1}\Lambda \oplus e_{v_1}\Lambda \oplus e_{v_1}\Lambda \\ P_4 &= e_{\tau(Cddba)}\Lambda \oplus e_{\tau(Cddd)} \oplus e_{\tau(Cbaaa)}\Lambda \oplus \ e_{\tau(Aaaa)}\Lambda \\ &= e_{v_1}\Lambda \oplus e_{v_2}\Lambda \oplus e_{v_2}\Lambda \oplus e_{v_1}\Lambda \oplus e_{v_1}\Lambda \\ P_5 &= e_{\tau(Cdddd)}\Lambda \oplus \ e_{\tau(Cdddb)}\Lambda \oplus e_{\tau(Cddbaa)} \oplus e_{\tau(Cdbacd)}\Lambda \\ &\oplus \ e_{\tau(Cbaaaa)}\Lambda \oplus \ e_{\tau(Aaaaa)}\Lambda \\ &= e_{v_2}\Lambda \oplus e_{v_1}\Lambda \oplus e_{v_1}\Lambda \oplus e_{v_2}\Lambda \oplus e_{v_1}\Lambda \oplus e_{v_1}\Lambda \end{aligned}$$

with the maps of generators given by

$$\partial_3 = \begin{bmatrix} c & 0 & a & 0 \\ d & b & 0 & 0 \\ 0 & 0 & cb & a \end{bmatrix}$$

$$\partial_4 = \begin{bmatrix} ba & d & 0 & 0 & 0 \\ 0 & c & ac+c & 0 & 0 \\ cb & 0 & 0 & a & 0 \\ 0 & 0 & 0 & cb & a \end{bmatrix}$$

$$\partial_5 = \begin{bmatrix} c & 0 & a & 0 & 0 & 0 \\ d & b & 0 & 0 & 0 & 0 \\ 0 & ba+b & 0 & d & 0 & 0 \\ 0 & 0 & cb & 0 & a & 0 \\ 0 & 0 & 0 & 0 & cb & a \end{bmatrix}$$

This resolution continues on indefinitely and can be computed rapidly for small n on a computer using the author's implementation in GAP. For the program code, see `http://math.arizona.edu/~pawloski/programs`.

Chapter 4
Implementations and Examples in GAP

We have given all of the algorithms needed to compute projective resolutions of simple A-modules for a finite dimensional algebra A. We have chosen to use the linear algebra techniques described after comparing timings of the linear algebra method of computing resolutions versus the Anick-Green method (see section 5.7). In addition, we have given the necessary theory and algorithms to compute the generators and relations of the Ext-algebra and cohomology ring of $\Bbbk G$ by computing in the basic algebra B. In this chapter, we give an expository description of the algorithms and how the theory is implemented in GAP. Throughout the chapter we use our running example of the basic algebra B generated from the group algebra of the symmetric group on four letters over $\mathbb{F}_2$. For more information on the data structures that are used see Appendix B. For all of the author's programs referred to in this dissertation written for GAP see `www.math.arizona.edu/~pawloski/programs`.

4.1 Cohomology and Ext

We now describe our fully automated program to compute a minimal set of generators and relations for $\dotplus_{k=0}^{n} \dotplus_{i,j} \operatorname{Ext}_B^k(S_i, S_j)$ for the Morita equivalent basic algebra B for a group algebra $\Bbbk G$. We shall only describe the procedure for computing

$$E(B) = \dotplus_{k=0}^{n} \dotplus_{i,j} \operatorname{Ext}_B^k(S_i, S_j)$$

as the cohomology ring is a special case where we simply compute $\dotplus_k \operatorname{Ext}_B^k(\Bbbk_B, \Bbbk_B)$ for the simple B-module $\Bbbk_B$ coming from the trivial $\Bbbk G$-module $\Bbbk$. In chapter 5 we present many of the results of the calculations from our implementation.

For the remainder of the section, we are in the following situation. Let B be the basic algebra of a group algebra $\Bbbk G$, where $\Bbbk$ is a splitting field. Assume that B is given in terms of a minimal set of generators (arrows and idempotents) and a $\Bbbk$-basis $\mathcal{B}_i$ for each PIM e_iB given as words in the generators with the matrices for the action of the generators on the basis elements. In our program we are supplied with this information from the results of Hoffman [Hof04]. As the $\Bbbk$-basis $\mathcal{B}$ for B is fixed, we may refer to the matrix of a linear map. Therefore whenever we refer to a linear map as a matrix, we mean with respect to this given basis $\mathcal{B}$.

The implementation begins by calculating a projective resolution of all simple modules. Cohomology and Ext-algebra elements are represented as chain maps on the computed pieces of the resolution. The products of elements are realized as compositions of the chain maps. The relations among the generators are obtained by rewriting the basis in terms of the generators and then applying the generators to the basis to see if we get another basis element or a relation.

The automated program for the calculation of Ext is called `ExtAlgebra`. It is a function of (basicalgebra,n). The program for the cohomology ring computation is `CohomologyRing`. It is a function of (basicalgebra,pimnumber,n) where pimnumber is the number of the PIM for the simple B-module coming from the trivial $\Bbbk G$-module. If you are only interested in the projective resolution of a module then you use the program `ProjectiveResolution`(basicalgebra,module,n).

Throughout, we continue our example of S_4 for illustrative purposes. However, for convenience we are going to relabel the original Ext-quiver

$$\begin{tikzcd} 1a \arrow[loop left, "1a1a1"] \arrow[r, shift left, "2a1a1"] & 2a \arrow[l, shift left, "1a2a1"] \arrow[loop right, "2a2a1"] \end{tikzcd}$$

as follows:

$$\begin{tikzcd} v_1 \arrow[loop left, "a"] \arrow[r, shift left, "c"] & v_2 \arrow[l, shift left, "b"] \arrow[loop right, "d"] \end{tikzcd}$$

We also take the liberty of switching back and forth between a vector and the polynomial that the vector represents. The vector $(0, 1, 0, 0, 0, 1)$ represents the sum of the second and the sixth word in PIM 1a, but we will often refer to it as $a + ac$. Now we describe how each step in the process is implemented.

Step 1: Minimal Resolution

Suppose that S_1 is a simple B-module. Our aim is to produce a minimal projective resolution for S_1. We begin with a minimal generating set for S_1 as a B-module. Each of the simple modules S_i is given as a vertex simple module corresponding to an idempotent e_i and we denote the corresponding PIM as $P(S_i)$. We know that the map $\varepsilon : P(S_1) \to S_1$ is given by quotienting by the radical. We also know from lemma 3.19 that the first two steps in the resolution are minimal. Therefore we have a minimal resolution $P_1 \to P_0 \to S_1$ that begins:

$$\bigoplus_{\substack{\text{arrows } a_j \\ o(a_j)=e_1}} e_{\tau(a_j)}B \xrightarrow{\partial_1} e_1 B \xrightarrow{\varepsilon} S_1 \to 0.$$

Recall from section 3.2.2 on page 93 that the map ∂_1 is just given by left multiplication by the arrows that originate from the PIM with simple S_1. We set up these maps and projective modules with the procedure `InitializeOmegaForBasicAlgebra`(basicalgebra,pimnumber).

```
gap> init:=InitializeOmegaForBasicAlgebra(basicalg,1);
 rec(rowblocks := [1], generators := [
   rec(blocks := [1],
     blockvector:=[[0*Z(2),Z(2)^0,0*Z(2),0*Z(2),0*Z(2),0*Z(2)]]),
   rec(blocks := [1],
     blockvector:=[[0*Z(2),0*Z(2),0*Z(2),0*Z(2),Z(2)^0,0*Z(2)]])],
  columnblocks := [ 1, 2 ] )
```

This represents the projective module P_1 and the sequence

$$P_1 \xrightarrow{\partial_1} P_0 \to S \to 0.$$

which is

$$e_{v_1}B \oplus e_{v_2}B \xrightarrow{\partial_1} e_{v_1}B$$

and the map ∂_1 is the beginning of the resolution. The map is given on the generators by $\partial_1(e_{v_1}, 0) = a$ and $\partial_1(0, e_{v_2}) = c$.

Next we construct the matrix for the $\Bbbk$-linear map ∂_1. This is easily constructed, because we have a basis for all of the PIMs written as products in the generators of the algebra B. Thus for each PIM $e_{\tau(a_j)}B$ we know that each of the basis elements $b \in e_{\tau(a_j)}B$ maps to $a_j \cdot b \in e_1B$. As we have images of all basis elements in a PIM, we simply record the matrix for this transformation. In `GAP` we get:

```
gap> hom;
 . 1 . . . .   # e_1 -> a
 . . . . . .   # a -> 0
 . . . 1 . .   # cb -> acb
 . . . . . .   # acb -> 0
 . . . . . 1   # c -> ac
 . . . . . .   # ac -> 0
 . . 1 . . .   # b -> cb
 . . . 1 . .   # ba -> cba = acb
 . . . . 1 .   # e_2 -> c
 . . . . . .   # d -> 0
 . . . . . .   # bc -> 0
```

The first 6 rows of the matrix represent the image vectors (in PIM 1a) of the basis elements of PIM 1a when applying the generator `a` on the left. The last 5 rows of the matrix represent the image vectors (in PIM 1a) of the basis elements of PIM 2a when applying the generator `c`. For example, the third row represents the mapping of `cb` to `acb` which we have recorded above.

The kernel of ∂_1, denoted $\Omega^2(S_1)$ as usual, is therefore the nullspace of the matrix of ∂_1. Computing the null space of a matrix is a standard operation in GAP using `NullspaceMat`(hom). The command `NullspaceMat` returns a $\Bbbk$-basis for the nullspace of this matrix:

```
gap> NullspaceMat(hom);
 . 1 . . . . . . . . .   # (a,0)
 . . . 1 . . . . . . .   # (acb,0)
 . . . . . 1 . . . . .   # (ac,0)
 . . 1 . . . . 1 . . .   # (cb,ba)
 . . . . . . . . . 1 .   # (0,d)
 . . . . . . . . . . 1   # (0,bc)
```

This is a partitioned matrix with the rows of the first six columns representing basis elements in PIM 1a and the last five columns are basis elements in PIM 2a. So for example, looking at the fourth row we see that `(cb,ba)` is in the kernel of $\partial_1 : e_{1a}B \oplus e_{2a}B \to e_{1a}B$.

Having computed $\Omega^2(S_1)$, the null space of ∂_1, we can compute $\mathrm{Rad}\,\Omega^2(S_1)$. We know that for a minimal resolution we have $\mathrm{Ker}\,\partial_{n-1} = \mathrm{Im}\,\partial_n \subseteq \mathrm{Rad}\,P_{n-1}$. Thus, the minimal generating set for $\Omega^2(S_1)$ is a basis $m_1, ..., m_s$ of a subspace $M \subseteq \Omega^2(S_1)$ that is complementary to $\mathrm{Rad}\,(\Omega^2(S_1))$. So $\Omega^2(S_1) = \mathrm{Span}_{\Bbbk}(m_1, ..., m_s) + \mathrm{Rad}\,(\Omega^2(S_1))$. To find a basis for the radical $\mathrm{Rad}\,(\Omega^2(S_1))$ we note that:

$$\mathrm{Rad}\left(\Omega^2(S_1)\right) = \left(\Omega^2(S_1)\right) \cdot \mathrm{Jac}\,B = \sum_{\text{arrows } a_i} \left(\Omega^2(S_1)\right) \cdot a_i,$$

where the arrows a_i are the nilpotent generators of the algebra B. Thus we multiply each of the basis elements b_i of $\Omega^2(S_1)$ by all of the nilpotent generators and take a basis $\mathcal{B}_{\mathrm{Rad}}$ for the $\Bbbk$-linear span of the products $a_j \cdot b_i$. We then extend $\mathcal{B}_{\mathrm{Rad}} = \{r_1, ..., r_m\}$ to a basis of $(\Omega^2(S_1))$.

As $\mathrm{Rad}\,\Omega^2(S_1) = \Omega^2(S_1) \cdot \mathrm{Jac}\,B$ and

$$\mathrm{Rad}\,\Omega^2(S_1) = \mathrm{Rad}\,\Omega^2(S_1) \cdot 1 = \mathrm{Rad}\,\Omega^2(S_1) \cdot \sum_i e_k = \bigoplus_k \mathrm{Rad}\,\Omega^2(S_1)\, e_k,$$

we can find a complementary basis to the basis of the radical by doing it for each idempotent e_k. This theoretical implication saves time and memory in our computation in `GAP`. We extend the basis $\{r_1, ..., r_m\}$ of $\mathrm{Rad}\left(\Omega^2\left(S_1\right)\right)$ to a basis of $\Omega^2\left(S\right)$ as follows. We do this by seeing whether the basis vectors in $\Omega^2\left(S\right)$ found using the `NullspaceMat` command are in $\mathrm{Span}_{\Bbbk}\{r_1, ..., r_m\}$. The GAP command for checking inclusion of a vector in a subspace is `IsContainedInSpan`(mutablebasis,vector). If the command returns false, we have found a minimal generator for the module $\Omega^2\left(S_1\right)$ and then add it to the basis for the radical of the kernel by the command `CloseMutableBasis`(mutablebasis,vector). We know that it is a minimal generator as each simple component in $\Omega^2\left(S_1\right) / \mathrm{Rad}\,\Omega^2\left(S_1\right)$ is 1-dimensional and therefore generated by one element. We do this procedure for each of the idempotents e_k of B until we have reached the proper dimension such that

$$\begin{aligned}\dim_{\Bbbk} \mathrm{Span}\left(\text{minimal generators of } \Omega\left(S_1\right) \cdot e_k\right) + \dim_{\Bbbk} \mathrm{Rad}\left(\Omega^2\left(S_1\right) \cdot e_k\right) \\ = \dim_{\Bbbk}\left(\Omega\left(S_1\right) e_k\right).\end{aligned}$$

After doing this for each idempotent e_k, we know that we have a minimal generating set for the kernel. After this step is complete, for storage purpose we unbind the basis of $\Omega^2\left(S_1\right)$ and only keep the minimal generators.

Continuing our example, we have computed the kernel of the homomorphism using our routine `KernelOfHom`:

```
gap> kernel:=KernelOfHom(basicalg,init);
 rec(
 rowblocks:=[1,2],
  basis:=[
   rec(blocks:=[1,2],blockvector:=[[0,1,0,0,0,0],[0,0,0,0,0]]),
   rec(blocks:=[1,2],blockvector:=[[0,0,0,1,0,0],[0,0,0,0,0]]),
   rec(blocks:=[1,2],blockvector:=[[0,0,0,0,0,1],[0,0,0,0,0]]),
```

```
   rec(blocks:=[1,2],blockvector:=[[0,0,1,0,0,0],[0,1,0,0,0]]),
   rec(blocks:=[1,2],blockvector:=[[0,0,0,0,0,0],[0,0,0,1,0]]),
   rec(blocks:=[1,2],blockvector:=[[0,0,0,0,0,0],[0,0,0,0,1]])
])
```

The next step is to remove the redundant generators. The GAP routine we have implemented is called `ModuleGeneratorsFromBasis`.

```
For PIM 1a, Rad(Omega)e_1 =
MutableBasis(GF(2),[[0,0,0,1,0,0,0,0,0,0,0]])
```

For us, this corresponds to the block vector `[[0,0,0,1,0,0],[0,0,0,0,0]]` (or $(acb, 0)$). We now keep the basis vectors for $\Omega^2(S_1)e_1$ that are not in $\operatorname{Rad}(\Omega^2(S_1)e_1)$. The vectors in $\Omega^2(S_1)e_1$ are the vectors in basis of the kernel that end in PIM 1a. From the basis of the kernel above from `KernelOfHom`, we see that the 1^{st}, 2^{nd}, and 4^{th} words end in PIM 1a. However, we do not consider the 4^{th} word as it is in $\operatorname{Rad}(\Omega^2(S_1))\, e_1$. We will keep both the 1^{st} and the 2^{nd} as they are linearly independent vectors. We then do the same for PIM 2a.

We next construct a projective cover $\omega_2 : P_2 \to \Omega^2(S)$. Recall that the projective covers are additive by Proposition 1.28.3 on page 43. We know that for each simple module S_1 we have corresponding projective cover $P(S_1)$ and therefore all we need to keep track of are where the minimal generators in $\Omega^2(S_1)$ begin and end and the image vector of the generator. Therefore, we can record ∂_2 as the list of vectors $\alpha_{i,j}$. That is, the output of the program for the construction of P_2 and ∂_2 consists of a record of the projective modules as a list of numbers such as [1,1,2,2,3]. This refers to the domain of the map as $e_1B \dotplus e_1B \dotplus e_2B \dotplus e_2B \dotplus e_3B$ (see section B.4 for more information). It also includes a list [1,1,2] for the range of the map. The other piece of data we record is a list of images of the idempotents e_i in the domain by storing the corresponding partitioned row vector.

The following is what we obtain in the computation that gets us to P_2 and ∂_2 in the resolution of S_4.

```
rowblocks:=[1,2],
  generators:=[
   rec(blocks:=[1,2], blockvector:=[[0,1,0,0,0,0],[0,0,0,0,0]]),
   rec(blocks:=[1,2], blockvector:=[[0,0,1,0,0,0],[0,1,0,0,0]]),
   rec(blocks:=[1,2], blockvector:=[[0,0,0,0,0,0],[0,0,0,1,0]])],
columnblocks:=[1,1,2])]
```

This tells us that $P_2 \xrightarrow{\partial_2} P_1$ in the minimal resolution is

$$e_1B \oplus e_1B \oplus e_2B \to e_1B \oplus e_2B$$

where $(e_1, 0, 0) \mapsto (a, 0)$, $(0, e_1, 0) \mapsto (cb, ba)$, and $(0, 0, e_2) \mapsto (0, d)$

We repeat the described process some n times. The result is a portion

$$P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow S \longrightarrow 0$$

of the minimal projective resolution of S.

Step 2: Chain Maps The next step is to find a minimal generating set for

$$\dot{+}_{k=0}^{n} \dot{+}_{i,j} \operatorname{Ext}_B^k (S_1, S_j) \, .$$

The main command for this procedure is called `ExtAlgebra`(basicalg,n) (respectively `CohomologyGenerators`). Recall that we are looking at the cohomology and Ext-algebra of simple modules and that we have minimal projective resolutions and thus the Yoneda products of the cohomology elements are given as compositions of the chain maps by Proposition 2.5.

Once we have computed the projective resolutions of the simple modules up to degree n, we know the dimension of $\operatorname{Ext}_B^r (S_i, S_j)$, $0 \leq r \leq n$, as a vector space over $\Bbbk$. This is simply the number of times the PIM e_jB corresponding to the simple module S_j appears in the resolution of S_i. With respect to our data, it is just the number of times the number of the corresponding simple module appears in `module.columnblocks`. The command `HomologyDims`(basicalg,resolution,simple,n) computes these dimensions.

We have described the procedure to find the minimal set of generators in Algorithm 2.4.1 on page 72. This entire process relies on computing chain maps. The calculation of a chain map is a straightforward application of linear algebra. The function is `ComputeChainMaps` (basicalg,projres1,projres2,degree,ende,map). Once again, the actual map between the projective modules is computed by knowing the images of the generators. Obtaining the images of the generators is a matter of solving a system of linear equations. That is, suppose for cohomology element ι in degree n we have computed the chain map to degree r. So in the diagram below,

$$\begin{tikzcd} \cdots \arrow[r] & P_{n+r+1} \arrow[r, "\partial_{n+r+1}"] \arrow[d, "\iota_{r+1}"] & P_{n+r} \arrow[r] \arrow[d, "\iota_r"] & \cdots \\ \cdots \arrow[r] & P'_{r+1} \arrow[r, "\partial'_{r+1}"] & P'_r \arrow[r] & \cdots \end{tikzcd} \tag{4.1}$$

for each idempotent e_i of P_{n+r+1}, we must solve the equation $\partial'_{r+1}(u) = \iota_r \partial_{n+r+1}(e_i)$ for an element u of P_{r+1}. This answer is not unique and any solution will do.

We now describe how to implement this algorithm into GAP in more detail. Recall we are in the situation in (4.1) and want to compute ι_{r+1} such that the diagram commutes. That is we want to compute a lift ι_{r+1} such that $\iota_r \circ \partial_{n+r+1} = \partial'_{r+1} \circ \iota_{r+1}$. We first compute the composition $\iota_r \circ \partial_{n+r+1}$ with `CompositionOfHoms`(basicalg, mod1, mod2). For each of the idempotents $e_i \in P_{n+r+1}$, we would like to consider all possible maps to the idempotents e_j of P_r. Therefore we can look at the basis of the projective module e_iB of P_{n+r+1} and look up which of these words in the generators of the algebra end in the PIM corresponding to idempotent e_j . The basis $\mathcal{B}$ is ordered by PIMs such that for each $b_1, b_2 \in \mathcal{B}_{e_iB}$ we have $\tau(b_1) \geq \tau(b_2)$. Therefore we may use the Cartan matrix in the entry `basicalg.cartan[i][j]` to determine which words in e_iB end in e_jB. We then apply the map ∂'_{r+1} to these words. To do this we use the low level routine `ApplyTreeToBlockVectorRestrictedToIdempotent`. We then record the vector that we returned for each of the idempotents in P_{n+r+1} as a matrix. Then for each of the images of the generators of $\iota_r \circ \partial_{n+r+1}$ we need to solve the corresponding system of equations. To do this we use the GAP routine `SolutionMat`(matrix,vector).

To illustrate this important procedure we give an example. Suppose we have $\eta_{1,2,1} \in \operatorname{Ext}^1_B(S_1, S_2)$ and $\gamma_{2,2,2} \in \operatorname{Ext}^2_B(S_2, S_2)$. We would like to compute $\gamma_{2,2,2} \cdot \eta_{1,2,1}$. We therefore are looking at the map:

$$\begin{tikzcd}
e_1B \oplus e_1B \oplus e_1B \oplus e_2B \arrow[r, "\partial_3"] \arrow[d, "\iota_2"] & e_1B \oplus e_1B \oplus e_2B \arrow[r, "\partial_2"] \arrow[d, "\iota_1"] & e_1B \oplus e_2B \arrow[r, "\partial_1"] \arrow[d, "\iota_0"] \arrow[dr, "\eta_{1,2,1}"] & e_1B \arrow[r, "\varepsilon"] & S_1 \\
e_1B \oplus e_2B \oplus e_2B \arrow[r, "d_2"] \arrow[d, "\gamma_{2,2,2}"] & e_1B \oplus e_2B \arrow[r, "d_1"] & e_2B \arrow[r, "\epsilon"] & S_2 \arrow[r] & 0 \\
S_2 & & & &
\end{tikzcd}$$

The map ι_0 is just the standard map below:

```
gap>iota0;
rec(columnblocks:=[1,2],rowblocks:=[2],
 generators:=[
   rec(blocks:=[2],blockvector:=[[0,0,0,0,0]]),
   rec(blocks:=[2],blockvector:=[[0,0,1,0,0]])])
```

We would like to lift to ι_1 such that the corresponding square commutes. The first thing that we do is to compute $\iota_0 \circ \partial_2$.

$$\begin{aligned}
\iota_0 \circ \partial_2 (e_1, 0, 0) &= \partial_2 (a, 0) = 0 \\
\iota_0 \circ \partial_2 (0, e_1, 0) &= \partial_2 (cb, ba) = ba \\
\iota_0 \circ \partial_2 (0, 0, e_2) &= \partial_2 (0, d) = d
\end{aligned}$$

In GAP this is:

```
gap>CompositionOfHoms(basicalg,resolution1[2],iota0);
rec(columnblocks:=[1,1,2],rowblocks:=[2],
  generators:=[
    rec(blocks:=[2],blockvector:=[[0,0,0,0,0]]),
    rec(blocks:=[2],blockvector:=[[0,1,0,0,0]]),
    rec(blocks:=[2],blockvector:=[[0,0,0,1,0]])])
```

The next step is to compute possible images of the map $d_1 \circ \iota_1$. We do this on a PIM by PIM basis. In the projective module $e_1B \oplus e_1B \oplus e_2B$ we first look at possible maps under ι_1 to $e_1B \oplus e_2B$. We consider the possibilities for $(e_1, 0, 0)$. To be a possibility we must consider all words γ that start in PIM e_1B and end in PIM e_1B, i.e. $\gamma \in e_1B \cap Be_1$. This information is conveniently stored in the Cartan matrix, $C_{1,1}$. We look at $C_{1,1} = 4$ and know there are four such words. They are e_1, a, cb, and acb, the first four entries in the basis information in the basic algebra for PIM 1a. We then apply the map d_1 to these words. The result is $e_1 \mapsto b$, $a \mapsto ba$, and $cb \mapsto 0$. Then we next consider the map under ι_1 from e_1B to e_2B, i.e., all words in PIM 2a (e_2B) that end in PIM 1a (e_1B). Looking up $C_{2,1}$ we see that the first two entries in PIM 2a satisfy this property. The words are b and ba. We then apply the map d_1 and see that both b and ba map to 0. We know that under $\iota_0 \circ \partial_2$, $(e_1, 0, 0) \mapsto 0$ and the first generator of ι_1 is clearly 0. Next, we know that $(0, e_1, 0) \mapsto ba$. We need to write this as a linear combination of the words that we have seen. We therefore want $\iota_1(0, e_1, 0) = a$ and obtain the generator

```
columnblocks:=[1,1,2]
rec(blocks:=[1,2],blockvector:=[[0,1,0,0,0,0],[0,0,0,0,0]])
```

The last thing that we must compute are the possible images of e_2. We look for both words that begin in e_1B and e_2B and end in e_2B. Words that start in PIM 1a and end in PIM 2a are c and ac. Applying the map d_2 we end up with $c \mapsto bc$

and $ac \mapsto bac = bc$. Words that start in PIM 2a and end in PIM 2a are e_2, d, and bc. The respective images under d_1 are $e_2 \mapsto d$, $d \mapsto bc$, and $bc \mapsto 0$. We know that $\iota_0 \circ \partial_2 (0, 0, e_2) = d$ and so clearly we must send $(0, 0, e_2)$ to $(0, e_2)$. The final result for ι_1 is:

```
gap>iota1;
rec(rowblocks:=[1,2],columnblocks:=[1,1,2],
 generators:=[
   rec(blocks:=[1,2],blockvector:=[[0,0,0,0,0,0],[0,0,0,0,0]]),
   rec(blocks:=[1,2],blockvector:=[[0,1,0,0,0,0],[0,0,0,0,0]]),
   rec(blocks:=[1,2],blockvector:=[[0,0,0,0,0,0],[0,0,1,0,0]])])
```

We now repeat the same process to lift ι_1 to ι_2. We have that

$$
\begin{aligned}
\iota_1 \circ \partial_3 (e_1, 0, 0, 0) &= \iota_1 (a, 0, 0) = 0 \\
\iota_1 \circ \partial_3 (0, e_1, 0, 0) &= \iota_1 (cb, a, 0) = aa = 0 \\
\iota_1 \circ \partial_3 (0, 0, e_1, 0) &= \iota_1 (0, 0, b) = b \\
\iota_1 \circ \partial_3 (0, 0, 0, e_2) &= \iota_1 (0, c, d) = ac + d
\end{aligned}
$$

We now do the same lifting process PIM by PIM. We first consider the words that start in PIM 1a and end in PIM 1a. They are: e_1, a, cb and acb. We apply the map d_2 to the 1$^{\text{st}}$ slot and end up with images b, ba, 0, and 0 respectively. Thus we know that we want $\iota_1 (0, 0, b) = b$ and so we map $(0, 0, e_1, 0)$ to $(0, 0, e_2)$. We lastly consider words starting in e_2B and ending in e_1B. These are b and ba. We apply d_2 to the second slot and get $acb + cb$ and acb and to the third slot and get 0 and acb. So we know that we have $\iota_2 (0, 0, e_1, 0) = (e_1, 0, 0)$. We also know that both $(e_1, 0, 0, 0)$ and $(e_1, 0, 0, 0)$ map to $(0, 0, 0)$.

We now repeat the process for words that start in PIM 1a and end in PIM 2a and also that start in PIM 2a and end in PIM 2a.

- Start in PIM 1a and end in PIM 2a: $\{c, ac\} \mapsto \{bc, bc\}$.
- Start in PIM 2a and end in PIM 2a: $\{e_2, d, bc\} \mapsto \{ac + c, d, 0\}$ for image of second slot of d_2.
- Start in PIM 2a and end in PIM 2a: $\{e_2, d, bc\} \mapsto \{c + d, bc, bc\}$ for image of third slot of d_2.

As we know that $\iota_1 \circ \partial_3 (0, 0, 0, e_2) = ac + d$, and we want the diagram to commute, we need a linear combination of the items above to give us this. We thus need $(ac + c) + (c + d) = ac + d$ and so we need to send $(0, 0, 0, e_2)$ to $(0, e_2, e_2)$. In GAP the record is:

```
gap>iota2;
rec(rowblocks:=[1,2,2],columnblocks:=[1,1,1,2],
 generators:=[
   rec(blocks:=[1,2,2],
       blockvector:=[[0,0,0,0,0,0],[0,0,0,0,0],[0,0,0,0,0]]),
   rec(blocks:=[1,2,2],
       blockvector:=[[0,0,0,0,0,0],[0,0,0,0,0],[0,0,0,0,0]]),
   rec(blocks:=[1,2,2],
       blockvector:=[[1,0,0,0,0,0],[0,0,0,0,0],[0,0,0,0,0]]),
   rec(blocks:=[1,2,2],
       blockvector:=[[0,0,0,0,0,0],[0,0,1,0,0],[0,0,1,0,0]])])
```

Our last stage in the above is at each stage of our lift, to apply all standard basis elements to map to all possible simple modules. First we gather some important data. We are looking to compose a generator $\eta_{1,2,1} \in \mathrm{Ext}^1_B (S_1, S_2)$ with all compatible

$\gamma_{2,1,1} \in \operatorname{Ext}^1_B(S_2, S_1)$ and $\gamma_{2,2,1} \in \operatorname{Ext}^1_B(S_2, S_2)$. We first note the dimensions of the vector spaces we are considering: $\operatorname{Dim}_{\Bbbk} \operatorname{Ext}^1_B(S_2, S_1) = 1$ and $\operatorname{Dim}_{\Bbbk} \operatorname{Ext}^1_B(S_2, S_2) = 1$. Therefore we compute both $\gamma_{2,1,1} \circ \iota_1$ and $\gamma_{2,2,1} \circ \iota_1$ and record the results as elements of $\operatorname{Ext}^2_B(S_1, S_1)$ and $\operatorname{Ext}^2_B(S_1, S_2)$ respectively. We then move on to the next level: $\operatorname{Dim}_{\Bbbk} \operatorname{Ext}^1_B(S_2, S_1) = 1$ and $\operatorname{Dim}_{\Bbbk} \operatorname{Ext}^1_B(S_2, S_2) = 2$. So we take $\gamma_{2,1,2} \in \operatorname{Ext}^2_B(S_2, S_1)$ and compute $\gamma_{2,1,2} \circ \iota_2$ which gives us $\gamma_{2,1,2}\eta_{1,2,1} \in \operatorname{Ext}^3_B(S_1, S_1)$. Similarly as the dimension of $Ext^1_B(S_2, S_2) = 2$ we take the standard basis $\gamma_{2,2,2}$ and $\rho_{2,2,2}$ and compute $\gamma_{2,2,2} \circ \iota_2$ and $\rho_{2,2,2} \circ \iota_2$. We end up with elements of $\operatorname{Ext}^3_B(S_1, S_2)$.

After we have completed finding the generators by using the chain map lifts, we have a record of all generators out to degree n and also all Yoneda compositions of the generators η_i and the standard basis of the vector space. The next thing we would like to do is to rewrite the standard basis in terms of products in the generators. This will give a nice basis in terms of finding all of the relations for the generators and giving a Gröbner basis presentation of the ideal of the generators. The standard approach is to compute all possible products of monomials up to degree n in the generators. Then the relations in degree n form a basis for the space of relations among the vectors of the monomials in $\Bbbk^s$. This is again the null space of the matrix whose rows are the vectors of the monomials. Computing the null space is a standard application of linear algebra. The next step would be to run a standard Buchberger algorithm to reduce this list of generators and have a Gröbner basis for the relations for later computational purposes. However as n grows the possible products in the generators gets out of hand pretty quickly. That is why we prefer the alternative Gröbner basis approach.

Step 3: Spin Up Basis in Generators: We have computed all of the generators and their products up to a given degree n. We have a list of generators $\{\eta_1, ..., \eta_m\}$ for the Ext-algebra $E(B)$ up to degree n. Our goal is to produce a graded $\Bbbk$-basis for $E(B)$. We first initialize the new basis $\mathcal{B} := \{\eta_1, ..., \eta_m\}$. For each $\eta_i \in \{\eta_1, ..., \eta_m\} = \mathcal{B}$ and each $r \in \{1, ..., m\}$ we compute the product $\eta_i\eta_r$. If $\eta_i\eta_r$ is not contained in the span of $\mathcal{B}$, then we append $\eta_i\eta_r$ to $\mathcal{B}$. We continue this procedure until it terminates and, i.e. we no longer find any new products. We know that this process is guaranteed to terminate as $\mathrm{Ext}^i_B(S_i, S_j)$ is finite for each i and we are only carrying out this procedure until we reach $i = n$. We know that we will find a basis because we know that we have a list of generators of the Ext-algebra. The algorithm that is used is Algorithm 2.4.2.

Step 4: Relations and Gröbner Basis We are given a basis for the algebra and a record of all of the generators on the basis. We wish to find all relations between the generators and present the ideal I that they generate as a Gröbner basis $\mathcal{G}$. As we have a basis given in terms of monomials in the generators and the action of all of the generators on this basis, we are in the situation we have had before to use our Alternate Gröbner Basis Algorithm 3.1.3. We use the same algorithm adapted to Ext-algebra and the routine is called `GrobnerBasisForExt`.

We now have a record of all products in the Ext-algebra, the generators, and relations so that we have an isomorphism between the Ext-algebra to a certain degree and the quotient of our given path algebra by the relations.

What remains to be seen is that n has been chosen large enough so that we have found all of the generators $\eta_1, ..., \eta_r$ and relations needed to have an isomorphism

$$E(\Bbbk G) \cong \Bbbk\langle \eta_1, ..., \eta_r\rangle / \langle \mathcal{G}\rangle.$$

This remains to be investigated as more theoretical results are needed in the case of a basic algebra.

Chapter 5
Results

In this chapter, we present the results of some of our calculations for the cohomology ring and Ext-algebra of various group algebras for the principal block of specific groups over various characteristics. Since we are only concerned with principal blocks, for the rest of this chapter, all results are only for the principal block of $\Bbbk G$. We use the notation from the Atlas of Finite Groups [CCN+85] to list each group.

The first section will contain a brief summary of which cohomology rings and Ext-algebras are previously known for specific group algebras in the literature.

5.1 Data Summary

There are not many sources where Ext-algebras have been recorded. The most notable is in Benson and Carlson [BC87] for group algebras. In [Gen01, GO02, Gen02, GK03, GK04], Generalov et al. give generators and relations of the Ext-algebra for a infinite families of dihedral algebras. One case includes groups with dihedral Sylow subgroups.

More results are known and have been published for cohomology rings. Most of these results such as the book by Adem and Milgram [AM04] are specifically done for cohomology in characteristic 2. Among the sporadic groups that have been completed for cohomology in characteristic 2 are M_{22} by Adem-Milgram [AM95], the cohomology of M_{23} by Milgram [Mil00], McL by Adem-Milgram [AM97], Ly by Adem et al. [AKMU98], and that of J_2, J_3 by Carlson-Maginnis-Milgram [CMM99]. For the Higman Sims group HS, the cohomology of the 2-Sylow subgroup was calculated in [ACKM01]. Cohomology rings such as that for the Held group He and M_{24} in characteristic 2 represent a new level of complexity and have not yet been determined. For classical groups over fields of finite characteristic see Priddy and Fiedorowicz

[FP78]. For the specific case of $\mathrm{SL}(2, p^n)$ see Carlson [Car83].

In addition to these results, theoretical results concerning the cohomology ring and Ext-algebras for group algebra $\Bbbk G$ in characteristic p where p divides the group order of G only once are well known. The cohomology ring in this setting is described in Green [Gre74] and the Ext-algebra case in Brown [Bro99]. We include our results for these cases mainly as verifications that our programs are running correctly.

The following tables contain the references for previously known cohomology rings and Ext-algebras that we have used to verify that our programs are working correctly. Note that, for the Ext-algebra computations, the generators and relations for the full Ext-algebra are not given in [BC87]. All that is supplied is $\mathrm{Ext}^*_{\Bbbk G}(S, S)$ for all of the simple modules in the principal block.

The following tables contain the references for previously known cohomology rings and Ext-algebras, as well as the page numbers for our results in this dissertation.

Group	Prime	Reference	Page
A_6	2	[AM04, pages 209-211]	136
A_7	2	[BC87, page 111]	137
A_8	2	[AM04, pages 209-211]	139
A_{10}	2	[AM04, pages 209-211]	142
S_4	2	[BC87, page 112]	144
S_6	2	[AM04, pages 203-206]	146
S_8	2	[AM04, pages 203-206]	148
M_{11}	2	[AM04, page 247]	152
M_{12}	2	[AM04, page 255]	154
J_1	2	[AM04, page 247]	156

Table 5.1. Some Known Cohomology Rings

5.2 Data Description

In the results we present, we refer to the computation of the cohomology ring and Ext-algebra. By this we mean that we have partially calculated the cohomology ring

Group	Prime	Reference	Page
A_6	3	[BC87, page 107]	136
A_7	2	[BC87, page 111]	137
S_4	2	[BC87, page 112]	144
M_{11}	2	[BC87, pages 97-107]	152
$L_3(3)$	2	[BC87, page 111]	170

TABLE 5.2. Some Known Ext-Algebras

and Ext-algebra up to a chosen degree n. For most of these groups, the complete description of the cohomology ring and Ext-algebra would be long and not useful to read. Instead, we include a list of generators for each. When small enough, we also include the Gröbner basis $\mathcal{G}$ for ideal of relations among the generators. For the cohomology ring we denote the generators as x_n where n is the degree of the generator. If there is more than one generator of a given degree, we use the next available letter in the alphabet. For the Ext-algebra we refer to the generators as $\eta_{i,j,k}$ which indicates $\eta_{i,j,k} \in \mathrm{Ext}^k(S_i, S_j)$. When more then one generator is found in $\mathrm{Ext}^k(S_i, S_j)$ we use another Greek letter such as $\xi_{i,j,k}$.

For denoting elements of a finite field, we use the notation $Z(p^d)$ to denote the generator of multiplicative group of the finite field with p^d elements. See the `GAP` webpage `http://www.gap-system.org/Manuals/doc/htm/ref/CHAP057.htm` for more information on how the specific generator is chosen.

For presenting the results of our cohomology ring calculations, we use the notation

$$H^*(G, \Bbbk) \cong \Bbbk[x_1, ..., x_m] / \langle \mathcal{G} \rangle$$

to mean the quotient of the graded-commutative polynomial ring where

$$x_i \cdot x_j = (-1)^{i \cdot j} x_j \cdot x_i.$$

Note that for some of our computations, we include the results of the Ext-algebra up to degree n which is smaller than the degree of the corresponding cohomology ring calculation. This is due to the fact that an Ext-algebra computation has many more computations than the cohomology ring as we need to compute for all possible pairs (S_i, S_j) versus just for $(\mathbb{k}, \mathbb{k})$.

5.3 Alternating Groups

5.3.1 A_4

The order of A_4 is $2^2 \cdot 3 = 12$.

Characteristic 2: For the splitting field $\mathbb{F}_4$ with degree of computation $n = 40$:

$$H^*(A_4, \mathbb{F}_4) \cong \mathbb{F}_4 [x_2, x_3, y_3] / \langle x_2^3 + x_3 y_3 \rangle.$$

The number of generators is 3 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 6 generators:

$$\eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,3,1},\ \eta_{3,1,1},\ \eta_{3,2,1},$$

with $\mathcal{G}$ the set of size 6:

$$\eta_{3,1,1}\eta_{1,3,1} + \eta_{2,1,1}\eta_{1,2,1},\ \eta_{3,2,1}\eta_{2,3,1} + \eta_{1,2,1}\eta_{2,1,1},\ \eta_{2,3,1}\eta_{3,2,1} + \eta_{1,3,1}\eta_{3,1,1},$$
$$\eta_{1,3,1}\eta_{3,1,1}\eta_{2,3,1} + \eta_{2,3,1}\eta_{1,2,1}\eta_{2,1,1},\ \eta_{1,2,1}\eta_{2,1,1}\eta_{3,2,1} + \eta_{3,2,1}\eta_{1,3,1}\eta_{3,1,1},$$
$$\eta_{2,1,1}\eta_{1,2,1}\eta_{3,1,1}\eta_{2,3,1} + \eta_{3,1,1}\eta_{2,3,1}\eta_{1,2,1}\eta_{2,1,1}.$$

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 100$:

$$H^*(A_4, \mathbb{F}_3) \cong \mathbb{F}_3 [x_1, x_2] / \langle x_1^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 2 generators:

$$\eta_{1,1,1},\ \eta_{1,1,2},$$

with $\mathcal{G}$ the set of size 2:

$$\eta_{1,1,1}^2,\ \eta_{1,1,1}\eta_{1,1,2} + 2 \cdot \eta_{1,1,2}\eta_{1,1,1}.$$

5.3.2 A_5

The order of A_5 is $2^2 \cdot 3 \cdot 5 = 60$.

Characteristic 2: For the splitting Field $\mathbb{F}_4$ with degree of computation $n = 100$:

$$H^*(A_5, \mathbb{F}_4) \cong \mathbb{F}_4[x_2, x_3, y_3] / \langle x_2^3 + x_3 y_3 \rangle.$$

The number of generators is 3 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 9 generators:

$$\eta_{1,1,2},\ \eta_{1,1,3},\ \xi_{1,1,3},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,3},\ \eta_{3,1,1},\ \eta_{3,3,3},$$

where $|\mathcal{G}| = 20$ and the largest relation found is of degree 6.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 100$:

$$H^*(A_5, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4] / \langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 4 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{2,1,1},\ \eta_{2,1,2},$$

with $\mathcal{G}$ the set of size 4:

$$\eta_{2,1,1}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,1},\ \eta_{2,1,1}\eta_{1,2,2} + 2 \cdot \eta_{2,1,2}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,2} + 2 \cdot \eta_{1,2,2}\eta_{2,1,1}.$$

Characteristic 5: For the splitting Field $= \mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(A_5, \mathbb{F}_5) \cong \mathbb{F}_5[x_3, x_4] / \langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 5 generators:

$$\eta_{1,1,4},\ \eta_{1,2,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,4},$$

with $\mathcal{G}$ the set of size 7:

$$\eta_{2,1,1}\eta_{1,2,1},\ \eta_{2,2,1}\eta_{2,2,1} + Z(5)^3 \cdot \eta_{1,2,1}\eta_{2,1,1},\ \eta_{1,2,1}\eta_{2,1,1}\eta_{2,2,1} + Z(5)^2 \cdot \eta_{2,2,1}\eta_{1,2,1}\eta_{2,1,1},$$
$$\eta_{2,1,1}\eta_{2,2,1}\eta_{1,2,1}\eta_{2,1,1},\ \eta_{1,2,1}\eta_{1,1,4} + Z(5) \cdot \eta_{2,2,4}\eta_{1,2,1},$$
$$\eta_{2,1,1}\eta_{2,2,4} + Z(5)^3 \cdot \eta_{1,1,4}\eta_{2,1,1},\ \eta_{2,2,1}\eta_{2,2,4} + Z(5)^2 \cdot \eta_{2,2,4}\eta_{2,2,1}.$$

5.3.3 A_6

The order of A_6 is $2^3 \cdot 3^2 \cdot 5 = 360$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with the degree of computation $n = 40$:

$$H^*(A_6, \mathbb{F}_2) \cong \mathbb{F}_2[x_2, x_3, y_3]/\langle x_3y_3\rangle.$$

The number of generators is 3 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 9 generators:

$$\eta_{1,1,2},\ \eta_{1,1,3},\ \xi_{1,1,3},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,3},\ \eta_{3,1,1},\ \eta_{3,3,3},$$

where $|\mathcal{G}| = 20$ and the largest relation found is of degree 6.

Characteristic 3: For the splitting field $\mathbb{F}_9$ with degree of computation $n = 30$:

$$H^*(A_6, \mathbb{F}_9) \cong \mathbb{F}_9[x_2, x_3, y_3, x_4, x_7, y_7, x_8, y_8]/\langle\mathcal{G}\rangle.$$

where $\mathcal{G}$ is the set:

$$x_2^2, x_2x_3, x_2y_3, x_2x_7, x_2y_7, x_3^2, x_3y_3 + Z(3^2)^3 \cdot x_2x_4, x_4^2x_8 + Z(3) \cdot y_8^2 + Z(3^2)^2 \cdot x_8y_8,$$
$$x_3x_7 + x_2x_8 + x_2y_8, x_3y_7 + Z(3^2)^6 \cdot x_2y_8, y_3^2, y_3x_7 + Z(3^2)^3 \cdot x_2y_8,$$
$$y_3y_7 + Z(3) \cdot x_2y_8, x_4x_7 + Z(3^2) \cdot y_3y_8 + x_3y_8 + Z(3^2) \cdot y_3x_8,$$
$$x_4y_7 + Z(3^2)^7 \cdot y_3y_8 + Z(3^2) \cdot x_3y_8, x_2x_4x_8 + Z(3^2)^5 \cdot x_7y_7, y_7^2, x_7^2,$$
$$y_3x_4x_8 + Z(3^2)^5 \cdot x_7y_8 + y_7y_8,$$
$$x_3x_4x_8 + Z(3^2)^5 \cdot y_7x_8 + Z(3^2)^7 \cdot x_7y_8 + Z(3^2)^5 \cdot y_7y_8.$$

The number of generators is 8 and $|\mathcal{G}| = 20$.

The Ext-algebra computation for $n = 30$ produces the following 26 generators:

$$\eta_{1,1,3},\ \eta_{1,1,8},\ \eta_{1,4,1},\ \eta_{2,2,3},\ \xi_{2,2,3},\ \eta_{2,2,4},\ \eta_{2,2,8},\ \xi_{2,2,8},\ \eta_{2,4,1},\ \xi_{2,4,1},\ \eta_{2,4,6},\ \xi_{2,4,6},\ \eta_{3,3,3},$$

$$\eta_{3,3,8},\ \eta_{3,4,1},\ \eta_{4,1,1},\ \eta_{4,2,1},\ \xi_{4,2,1},\ \eta_{4,2,6},\ \xi_{4,2,6},\ \eta_{4,3,1},\ \eta_{4,4,3}, \xi_{4,4,3},\ \eta_{4,4,4},\ \eta_{4,4,8},\ \xi_{4,4,8},$$

where $|\mathcal{G}| = 185$ and the largest relation found is of degree 30.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(A_6, \mathbb{F}_5) \cong \mathbb{F}_5[x_3, x_4]/\langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 5 generators:

$$\eta_{1,1,4},\ \eta_{1,2,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,4},$$

with $\mathcal{G}$ the set of size 7:

$$\eta_{2,1,1}\eta_{1,2,1},\ \eta_{2,2,1}\eta_{2,2,1} + \eta_{1,2,1}\eta_{2,1,1},\ \eta_{1,2,1}\eta_{2,1,1}\eta_{2,2,1} + Z(5)^2 \cdot \eta_{2,2,1}\eta_{1,2,1}\eta_{2,1,1},$$
$$\eta_{2,1,1}\eta_{2,2,1}\eta_{1,2,1}\eta_{2,1,1},\ \eta_{1,2,1}\eta_{1,1,4} + \eta_{2,2,4}\eta_{1,2,1},$$
$$\eta_{2,1,1}\eta_{2,2,4} + \eta_{1,1,4}\eta_{2,1,1},\ \eta_{2,2,1}\eta_{2,2,4} + Z(5)^2 \cdot \eta_{2,2,4}\eta_{2,2,1}.$$

5.3.4 A_7

The order of A_7 is $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 40$:

$$H^*(A_7, \mathbb{F}_2) \cong \mathbb{F}_2[x_2, x_3, y_3]/\langle x_3 y_3 \rangle.$$

The number of generators is 3 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 8 generators:

$$\eta_{1,1,2},\ \eta_{1,1,3},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{3,1,1},\ \eta_{3,3,3},$$

where $|\mathcal{G}| = 33$ and the largest relation found is of degree 40.

Characteristic 3: For the splitting field $\mathbb{F}_9$ with degree of computation $n = 30$:

$$H^*(A_7, \mathbb{F}_9) \cong \mathbb{F}_9[x_2, x_3, y_3, x_4, x_7, y_7, x_8, y_8] / \langle \mathcal{G} \rangle.$$

The number of generators is 8 and $|\mathcal{G}| = 20$. The set $\mathcal{G}$ is identical to that of $\mathbb{F}_9 A_6$.

The Ext-algebra computation for $n = 30$ produces the following 25 generators:

$$\eta_{1,1,3},\ \xi_{1,1,3},\ \eta_{1,1,4},\ \eta_{1,1,8},\ \xi_{1,1,8},\ \eta_{1,2,1},\ \xi_{1,2,1},\ \eta_{1,3,4},\ \eta_{1,4,4},\ \eta_{2,1,1},\ \xi_{2,1,1},\ \eta_{2,3,1},\ \eta_{2,3,2},$$
$$\eta_{2,4,1},\ \eta_{2,4,2},\ \eta_{3,1,4},\ \eta_{3,2,1},\ \eta_{3,2,2},\ \eta_{3,4,3},\ \eta_{3,4,4},\ \eta_{4,1,4},\ \eta_{4,2,1},\ \eta_{4,2,2},\ \eta_{4,3,3},\ \eta_{4,3,4},$$

where $|\mathcal{G}| = 240$ and the largest relation found is of degree 30.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(A_7, \mathbb{F}_5) \cong \mathbb{F}_5[x_7, x_8] / \langle x_7^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 10 generators:

$$\eta_{1,3,1},\ \eta_{1,4,4},\ \eta_{2,3,1},\ \eta_{2,3,4},\ \eta_{2,4,1},\ \eta_{3,1,1},\ \eta_{3,2,1},\ \eta_{3,2,4},\ \eta_{4,1,4},\ \eta_{4,2,1},$$

with $\mathcal{G}$ the set of size 15:

$$\eta_{3,1,1}\eta_{1,3,1},\ \eta_{4,2,1}\eta_{2,4,1} + Z(5)^3 \cdot \eta_{3,2,1}\eta_{2,3,1},\ \eta_{2,3,1}\eta_{3,2,1} + \eta_{1,3,1}\eta_{3,1,1},$$
$$\eta_{2,4,1}\eta_{4,2,1},\ \eta_{2,4,1}\eta_{3,2,1}\eta_{2,3,1},\ \eta_{3,2,1}\eta_{2,3,1}\eta_{4,2,1},\ \eta_{3,2,1}\eta_{1,3,1}\eta_{3,1,1}\eta_{2,3,1},$$
$$\eta_{2,4,1}\eta_{3,2,1}\eta_{1,3,1}\eta_{3,1,1},\ \eta_{1,3,1}\eta_{3,1,1}\eta_{2,3,1}\eta_{4,2,1},\ \eta_{4,2,1}\eta_{1,4,4} + Z(5) \cdot \eta_{3,2,4}\eta_{1,3,1},$$
$$\eta_{3,1,1}\eta_{2,3,4} + Z(5)^2 \cdot \eta_{4,1,4}\eta_{2,4,1},$$
$$\eta_{3,2,1}\eta_{2,3,4} + Z(5)^2 \cdot \eta_{3,2,4}\eta_{2,3,1},\ \eta_{2,3,1}\eta_{3,2,4} + Z(5)^2 \cdot \eta_{2,3,4}\eta_{3,2,1},$$
$$\eta_{2,4,1}\eta_{3,2,4} + Z(5)^2 \cdot \eta_{1,4,4}\eta_{3,1,1},\ \eta_{1,3,1}\eta_{4,1,4} + Z(5)^3 \cdot \eta_{2,3,4}\eta_{4,2,1}.$$

Characteristic 7: For the splitting Field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(A_7, \mathbb{F}_7) \cong \mathbb{F}_7[x_5, x_6]/\langle x_5^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 8 generators:

$$\eta_{1,1,6},\ \eta_{1,3,1},\ \eta_{2,2,1},\ \eta_{2,2,6},\ \eta_{2,3,1},\ \eta_{3,1,1},\ \eta_{3,2,1},\ \eta_{3,3,6},$$

with $\mathcal{G}$ the set of size 14:

$$\eta_{3,1,1}\eta_{1,3,1},\ \eta_{3,2,1}\eta_{2,3,1} + Z(7)\cdot\eta_{2,2,1}\eta_{2,2,1},\ \eta_{2,3,1}\eta_{3,2,1} + Z(7)\cdot\eta_{1,3,1}\eta_{3,1,1},$$
$$\eta_{1,3,1}\eta_{3,1,1}\eta_{2,3,1} + Z(7)^3\cdot\eta_{2,3,1}\eta_{2,2,1}\eta_{2,2,1},\ \eta_{2,2,1}\eta_{2,2,1}\eta_{3,2,1} + Z(7)^3\cdot\eta_{3,2,1}\eta_{1,3,1}\eta_{3,1,1},$$
$$\eta_{3,1,1}\eta_{2,3,1}\eta_{2,2,1}\eta_{2,2,1},\ \eta_{2,3,1}\eta_{2,2,1}\eta_{2,2,1}\eta_{2,2,1}\eta_{2,2,1},\ \eta_{2,2,1}\eta_{2,2,1}\eta_{2,2,1}\eta_{2,2,1}\eta_{2,2,1}\eta_{2,2,1},$$
$$\eta_{1,3,1}\eta_{1,1,6} + Z(7)^5\cdot\eta_{3,3,6}\eta_{1,3,1},\ \eta_{2,2,1}\eta_{2,2,6} + Z(7)^3\cdot\eta_{2,2,6}\eta_{2,2,1},$$
$$\eta_{3,1,1}\eta_{3,3,6} + Z(7)\cdot\eta_{1,1,6}\eta_{3,1,1}, \eta_{3,2,1}\eta_{3,3,6} + Z(7)^3\cdot\eta_{2,2,6}\eta_{3,2,1},$$
$$\eta_{3,1,1}\eta_{2,3,1}\eta_{2,2,1}\eta_{3,2,1}\eta_{1,3,1}\eta_{3,1,1}, \eta_{2,3,1}\eta_{2,2,6} + Z(7)^3\cdot\eta_{3,3,6}\eta_{2,3,1}.$$

5.3.5 A_8

The order of A_8 is $2^6 \cdot 3^2 \cdot 5 \cdot 7 = 20160$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 14$:

$$H^*(A_8, \mathbb{F}_2) \cong \mathbb{F}_2[x_2, x_3, y_3, x_4, x_5, x_6, y_6, x_7, y_7]/\langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is the set:

$$x_3y_3, x_3y_7 + x_2x_3x_5, y_3x_5, y_3x_6, y_3y_6, y_3x_7, y_3y_7, x_2^2x_5 + x_2y_7,$$
$$x_5^2 + x_2^2x_6 + x_2^2y_6 + x_2^2x_3^2 + x_2x_3x_5 + x_3x_7, x_5x_7 + x_2^2x_3x_5 + x_3x_4x_5,$$
$$x_5y_7 + x_2^3x_6 + x_2^3y_6 + x_2^3x_3^2 + x_2^2x_3x_5 + x_2x_3x_7,$$
$$x_2x_3x_4 + x_2x_7 + x_2^3x_3, x_2x_4y_7 + x_2^2x_4x_5,$$
$$x_2x_4x_7 + x_2^3x_7 + x_2^5x_3 + x_2x_3x_4^2,$$
$$x_6y_6 + x_2^3x_6 + x_2^3y_6 + x_2^3x_3^2 + x_2^2x_3x_5 + x_2x_3x_7 + x_2x_4x_6 + x_2x_4y_6 + x_3x_4x_5,$$
$$x_6y_7 + x_2x_5x_6, y_6x_7 + x_2^2x_3y_6 + x_3x_4y_6,$$
$$x_3^2x_4 + x_2^2x_3^2 + x_3x_7, x_7y_7 + x_2^3x_3x_5 + x_2x_3x_4x_5,$$
$$x_3x_4x_7 + x_2^4x_3^2 + x_2^2x_3x_7 + x_3^2x_4^2.$$

The number of generators is 9 and $|\mathcal{G}| = 20$.

The Ext-algebra computation for $n = 8$ produces 42 generators where the largest generator found is of degree 8 and $|\mathcal{G}| = 380$.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 30$:

$$H^*(A_8, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4, x_7, x_8]/\langle x_3^2, x_7^2\rangle.$$

The number of generators is 4 and $|\mathcal{G}| = 2$.

The Ext-algebra computation for $n = 30$ produces the following 20 generators:

$$\eta_{1,3,1},\ \eta_{1,3,2},\ \eta_{1,4,1},\ \eta_{2,3,1},\ \eta_{2,3,2},\ \eta_{2,4,1},\ \eta_{3,1,1},\ \eta_{3,1,2},\ \eta_{3,2,1}, \eta_{3,2,2},$$
$$\eta_{3,5,1},\ \eta_{3,5,2},\ \eta_{4,1,1},\ \eta_{4,2,1},\ \eta_{4,4,3},\ \eta_{4,4,8},\ \eta_{4,5,2},\ \eta_{5,3,1},\ \eta_{5,3,2},\ \eta_{5,4,2},$$

where $|\mathcal{G}| = 97$ and the largest relation found is of degree 30.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(A_8, \mathbb{F}_5) \cong \mathbb{F}_5[x_7, x_8]/\langle x_7^2\rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 10 generators:

$$\eta_{1,3,1},\ \eta_{1,4,4},\ \eta_{2,3,1},\ \eta_{2,3,4},\ \eta_{2,4,1},\ \eta_{3,1,1},\ \eta_{3,2,1},\ \eta_{3,2,4},\ \eta_{4,1,4},\ \eta_{4,2,1},$$

where $|\mathcal{G}| = 15$ and the largest relation found is of degree 5.

Characteristic 7 For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(A_8, \mathbb{F}_7) \cong \mathbb{F}_7[x_5, x_6]/\langle x_5^2\rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 8 generators:

$$\eta_{1,1,6},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,6},\ \eta_{3,1,1},\ \eta_{3,3,6},$$

where $|\mathcal{G}| = 14$ and the largest relation found is of degree 7.

5.3.6 A_9

The order of A_9 is $2^6 \cdot 3^4 \cdot 5 \cdot 7 = 181440$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 14$:

$$H^*(A_9, \mathbb{F}_2) \cong \mathbb{F}_2[x_2, x_3, y_3, x_4, x_5, x_6, y_6, x_7, y_7]/\langle\mathcal{G}\rangle.$$

The number of generators is 9 and $|\mathcal{G}| = 20$

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 20$:

$$H^*(A_9, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, y_3, x_4, y_4, x_7, y_7, x_8, y_8, z_8, x_9, x_{11}, x_{12}, y_{12}, x_{13}, x_{17}, x_{18}]/\langle\mathcal{G}\rangle.$$

The number of generators is 16 and $|\mathcal{G}| = 55$.

The Ext-algebra computation for $n = 15$ produces 74 generators where the largest generator found is of degree 15 and $|\mathcal{G}| = 1434$.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(A_9, \mathbb{F}_5) \cong \mathbb{F}_5 [x_7, x_8] / \langle x_7^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 10 generators:

$$\eta_{1,2,1},\ \eta_{1,2,4},\ \eta_{1,4,1},\ \eta_{2,1,1},\ \eta_{2,1,4},\ \eta_{2,3,1},\ \eta_{3,2,1},\ \eta_{3,4,4},\ \eta_{4,1,1},\ \eta_{4,3,4},$$

where $|\mathcal{G}| = 16$ and the largest relation found is of degree 5.

Characteristic 7 For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(A_9, \mathbb{F}_7) \cong \mathbb{F}_7 [x_{11}, x_{12}] / \langle x_{11}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 16 generators:

$$\eta_{1,3,1},\ \eta_{1,3,6},\ \eta_{1,5,1},\ \eta_{2,3,1},\ \eta_{2,5,6},\ \eta_{2,6,1},\ \eta_{3,1,1},\ \eta_{3,1,6},$$

$$\eta_{3,2,1}, \eta_{4,5,1},\ \eta_{4,6,6},\ \eta_{5,1,1},\ \eta_{5,2,6},\ \eta_{5,4,1},\ \eta_{6,2,1},\ \eta_{6,4,6},$$

where $|\mathcal{G}| = 32$ and the largest relation found is of degree 7.

5.3.7 A_{10}

The order of A_{10} is $2^7 \cdot 3^4 \cdot 5^2 \cdot 7 = 1814400$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 12$:

$$H^*(A_{10}, \mathbb{F}_2) \cong \mathbb{F}_2 [x_2, x_3, y_3, x_4, x_5, y_5, y_6, x_7] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is given by:

$$x_3x_7 + x_2^2x_3^2 + x_2x_3x_5, y_3x_5 + x_2x_3y_3 + x_3y_5, y_3x_6,$$
$$y_3x_7 + x_2x_3y_5, x_5y_5 + x_2^2x_3y_3 + x_3y_3x_4, y_5x_6, y_5x_7 + x_2^3x_3y_3 + x_2^2x_3y_5 + x_2x_3y_3x_4,$$
$$x_5^2 + x_2^2x_6 + x_2^2x_3^2 + x_2x_3x_5 + x_3^2x_4,$$
$$x_5x_7 + x_2^3x_6 + x_2^3x_3^2 + x_2x_3^2x_4, x_2^3x_3 + x_2x_7 + x_2^2x_5.$$

The number of generators is 8 and $|\mathcal{G}| = 10$.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 20$:

$$H^*(A_{10}, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, y_3, x_4, y_4, x_7, y_7, x_8, y_8, z_8, x_9, x_{11}, x_{12}, y_{12}, x_{13}, x_{17}, x_{18}]/\langle \mathcal{G} \rangle.$$

The number of generators is 16 and $|\mathcal{G}| = 55$.

The Ext-algebra computation for $n = 15$ produces 69 generators where the largest generator found is of degree 15 and $|\mathcal{G}| = 1604$.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 40$:

$$H^*(A_{10}, \mathbb{F}_5) \cong \mathbb{F}_5[x_6, x_7, y_7, x_8, x_{15}, y_{15}, x_{16}, y_{16}]/\langle \mathcal{G} \rangle.$$

The number of generators is 8 and $|\mathcal{G}| = 20$.

The Ext-algebra computation for $n = 20$ produces 100 generators where the largest generator found is of degree 16 and $|\mathcal{G}| = 1177$.

Characteristic 7 For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(A_{10}, \mathbb{F}_7) \cong \mathbb{F}_7[x_{11}, x_{12}]/\langle x_{11}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 16 generators:

$$\eta_{1,2,1},\ \eta_{1,4,6},\ \eta_{2,1,1},\ \eta_{2,3,1},\ \eta_{2,6,6},\ \eta_{3,2,1},\ \eta_{3,5,1},\ \eta_{3,5,6},$$

$$\eta_{4,1,6}, \eta_{4,6,1}, \eta_{5,3,1}, \eta_{5,3,6},\ \eta_{5,6,1},\ \eta_{6,2,6},\ \eta_{6,4,1},\ \eta_{6,5,1},$$

where $|\mathcal{G}| = 27$ and the largest relation found is of degree 7.

5.4 Symmetric Groups

5.4.1 S_4

The order of S_4 is $2^3 \cdot 3 = 24$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 100$:

$$H^*(S_4, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, x_3] / \langle x_1 x_3 \rangle.$$

The number of generators is 3 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 5 generators:

$$\eta_{1,1,1},\ \eta_{1,1,2},\ \eta_{1,2,1},\ \eta_{2,1,1},\ \eta_{2,2,1},$$

where $|\mathcal{G}| = 26$ and the largest relation found is of degree 40.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 100$:

$$H^*(S_4, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4] / \langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 4 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{2,1,1},\ \eta_{2,1,2},$$

where the set $\mathcal{G}$ is:

$$\eta_{2,1,1}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,1},\ \eta_{2,1,1}\eta_{1,2,2} + 2 \cdot \eta_{2,1,2}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,2} + 2 \cdot \eta_{1,2,2}\eta_{2,1,1}.$$

5.4.2 S_5

The order of S_5 is $2^3 \cdot 3 \cdot 5 = 120$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 40$:

$$H^*(S_5, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, x_3] / \langle x_1 x_3 \rangle.$$

The number of generators is 3 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 6 generators:

$$\eta_{1,1,1},\ \eta_{1,1,2},\ \eta_{1,1,3},\ \eta_{1,2,1},\ \eta_{2,1,1},\ \eta_{2,2,3},$$

where $|\mathcal{G}| = 29$ and the largest relation found is of degree 40.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 100$:

$$H^*(S_5, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4] / \langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 4 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{2,1,1},\ \eta_{2,1,2},$$

where $\mathcal{G}$ is also the same as in $\mathbb{F}_3 S_4$. We have an isomorphism of Ext-algebras to degree 100.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(S_5, \mathbb{F}_5) \cong \mathbb{F}_5[x_7, x_8] / \langle x_7^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 10 generators:

$$\eta_{1,2,1},\ \eta_{1,3,4},\ \eta_{2,1,1},\ \eta_{2,4,1},\ \eta_{2,4,4},\ \eta_{3,1,4},\ \eta_{3,4,1},\ \eta_{4,2,1},\ \eta_{4,2,4},\ \eta_{4,3,1},$$

with $\mathcal{G}$ the set of size 15:

$$\begin{aligned}
\eta_{2,1,1}\eta_{1,2,1},\ \eta_{4,2,1}\eta_{2,4,1} + Z(5)\cdot\eta_{1,2,1}\eta_{2,1,1},\ \eta_{4,3,1}\eta_{3,4,1},\ \eta_{3,4,1}\eta_{4,3,1} + \eta_{2,4,1}\eta_{4,2,1},\\
\eta_{2,4,1}\eta_{4,2,1}\eta_{3,4,1}, \eta_{4,3,1}\eta_{2,4,1}\eta_{4,2,1},\ \eta_{4,3,1}\eta_{2,4,1}\eta_{1,2,1}\eta_{2,1,1},\\
\eta_{1,2,1}\eta_{2,1,1}\eta_{4,2,1}\eta_{3,4,1},\ \eta_{2,4,1}\eta_{1,2,1}\eta_{2,1,1}\eta_{4,2,1},\ \eta_{3,4,1}\eta_{1,3,4} + Z(5)^2\cdot\eta_{2,4,4}\eta_{1,2,1},\\
\eta_{4,2,1}\eta_{2,4,4} + Z(5)^3\cdot\eta_{4,2,4}\eta_{2,4,1},\ \eta_{4,3,1}\eta_{2,4,4} + Z(5)^3\cdot\eta_{1,3,4}\eta_{2,1,1},\\
\eta_{1,2,1}\eta_{3,1,4} + Z(5)\cdot\eta_{4,2,4}\eta_{3,4,1},\\
\eta_{2,1,1}\eta_{4,2,4} + Z(5)^2\cdot\eta_{3,1,4}\eta_{4,3,1},\ \eta_{2,4,1}\eta_{4,2,4} + Z(5)\cdot\eta_{2,4,4}\eta_{4,2,1}.
\end{aligned}$$

5.4.3 S_6

The order of S_6 is $2^4 \cdot 3^2 \cdot 5 = 720$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 20$:

$$H^*(S_6, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, x_3, y_3] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is

$$x_1^6 + x_3y_3 + x_1x_2x_3 + x_1^3x_3 + x_1^4x_2 + x_1^3y_3$$

The number of generators is 4 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 20$ produces the following 12 generators:

$$\eta_{1,1,1},\ \eta_{1,1,2},\ \eta_{1,1,3},\ \xi_{1,1,3},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,3},\ \eta_{3,1,1},\ \eta_{3,3,1},\ \eta_{3,3,3},$$

where $|\mathcal{G}| = 131$ and the largest relation found is of degree 20.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 50$:

$$H^*(S_6, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4, x_7, x_8] / \langle x_3^2, x_7^2 \rangle.$$

The number of generators is 4 and $|\mathcal{G}| = 2$.

The Ext-algebra computation for $n = 30$ produces the following 42 generators:

$$\eta_{1,1,3},\ \eta_{1,1,4},\ \eta_{1,1,8},\ \eta_{1,2,3},\ \eta_{1,2,8},\ \eta_{1,3,1},\ \eta_{1,3,6},\ \eta_{1,5,1},\ \eta_{1,5,6}, \eta_{2,1,3},\ \eta_{2,1,8},$$

$$\eta_{2,2,3},\ \eta_{2,2,4},\ \eta_{2,2,8},\ \eta_{2,3,1},\ \eta_{2,3,6},\ \eta_{2,5,1},\ \eta_{2,5,6},\ \eta_{3,1,1},\ \eta_{3,1,6},\ \eta_{3,2,1},\ \eta_{3,2,6},$$

$$\eta_{3,3,3},\ \eta_{3,3,4},\ \eta_{3,3,8},\ \eta_{3,4,1},\ \eta_{3,5,3},\ \eta_{3,5,8},\ \eta_{4,3,1},\ \eta_{4,4,3},\ \eta_{4,4,8},\ \eta_{4,5,1},$$

$$\eta_{5,1,1},\ \eta_{5,1,6},\ \eta_{5,2,1},\ \eta_{5,2,6},\ \eta_{5,3,3},\ \eta_{5,3,8},\ \eta_{5,4,1},\ \eta_{5,5,3},\ \eta_{5,5,4},\ \eta_{5,5,8},$$

where $|\mathcal{G}| = 304$ and the largest relation found is of degree 16.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(S_6, \mathbb{F}_5) \cong \mathbb{F}_5[x_7, x_8] / \langle x_7^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 10 generators:

$$\eta_{1,2,1},\ \eta_{1,3,4},\ \eta_{2,1,1},\ \eta_{2,4,1},\ \eta_{2,4,4},\ \eta_{3,1,4},\ \eta_{3,4,1},\ \eta_{4,2,1},\ \eta_{4,2,4},\ \eta_{4,3,1},$$

where $|\mathcal{G}| = 15$ and the largest relation found is of degree 5.

5.4.4 S_7

The order of S_7 is $2^4 \cdot 3^2 \cdot 5 \cdot 7 = 5040$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 14$:

$$H^*(S_7, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, x_3, y_3] / \langle x_1x_2y_3 + x_3y_3 \rangle.$$

The number of generators is 4 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 10$ produces the following 11 generators:

$$\eta_{1,1,1},\ \eta_{1,1,2},\ \eta_{1,1,3},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \xi_{2,2,1},\ \eta_{3,1,1}, \eta_{3,3,1},\ \eta_{3,3,3},$$

where $|\mathcal{G}| = 81$ and the largest relation found is of degree 10.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 40$:

$$H^*(S_7, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4, x_7, x_8] / \langle x_3^2, x_7^2 \rangle.$$

The number of generators is 4 and $|\mathcal{G}| = 2$.

The Ext-algebra computation for $n = 20$ produces the following 32 generators:

$$\eta_{1,1,3},\ \eta_{1,1,4},\ \eta_{1,1,8},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{1,4,3},\ \eta_{1,4,8},\ \eta_{1,5,4},\ \eta_{2,1,1}, \eta_{2,4,1},\ \eta_{2,5,1},$$

$$\eta_{2,5,2},\ \eta_{3,1,1},\ \eta_{3,4,1},\ \eta_{3,5,1},\ \eta_{3,5,2},\ \eta_{4,1,3},\ \eta_{4,1,8},\ \eta_{4,2,1},\ \eta_{4,3,1},\ \eta_{4,4,3},\ \eta_{4,4,4},$$

$$\eta_{4,4,8},\ \eta_{4,5,4},\ \eta_{5,1,4},\ \eta_{5,2,1},\ \eta_{5,2,2},\ \eta_{5,3,1},\ \eta_{5,3,2},\ \eta_{5,4,4},\ \eta_{5,5,3},\ \eta_{5,5,4},$$

where $|\mathcal{G}| = 266$ and the largest relation found is of degree 20.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(S_7, \mathbb{F}_5) \cong \mathbb{F}_5[x_7, x_8] / \langle x_7^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 10 generators:

$$\eta_{1,2,1},\ \eta_{1,4,4},\ \eta_{2,1,1},\ \eta_{2,3,1},\ \eta_{2,3,4},\ \eta_{3,2,1},\ \eta_{3,2,4},\ \eta_{3,4,1},\ \eta_{4,1,4},\ \eta_{4,3,1},$$

where $|\mathcal{G}| = 15$ and the largest relation found is of degree 5.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(S_7, \mathbb{F}_7) \cong \mathbb{F}_7[x_{11}, x_{12}] / \langle x_{11}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 16 generators:

$$\eta_{1,2,1},\ \eta_{1,2,6},\ \eta_{1,5,1},\ \eta_{2,1,1},\ \eta_{2,1,6},\ \eta_{2,6,1},\ \eta_{3,4,6},\ \eta_{3,5,1},$$
$$\eta_{4,3,6},\ \eta_{4,6,1},\ \eta_{5,1,1},\ \eta_{5,3,1},\ \eta_{5,6,6},\ \eta_{6,2,1},\ \eta_{6,4,1},\ \eta_{6,5,6},$$

where $|\mathcal{G}| = 32$ and the largest relation found is of degree 7.

5.4.5 S_8

The order of S_8 is $2^7 \cdot 3^2 \cdot 5 \cdot 7 = 40320$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 12$:

$$H^*(S_7, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, x_3, y_3, x_4, x_5, x_6, x_7] / \langle \mathcal{G} \rangle.$$

The number of generators is 8 and $|\mathcal{G}| = 14$ with largest relation of degree 12.

The Ext-algebra computation for $n = 6$ produces the following 35 generators:

$$\eta_{1,1,2},\ \eta_{1,1,3},\ \xi_{1,1,3},\ \eta_{1,2,1},\ \eta_{1,5,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,2},\ \eta_{2,2,3},\ \xi_{2,2,3},\ \eta_{2,2,4},\ \eta_{2,2,5},$$
$$\eta_{2,3,1},\ \eta_{2,4,1},\ \eta_{3,2,1},\ \eta_{3,3,1},\ \eta_{3,3,2},\ \eta_{3,3,3},\ \eta_{3,3,4},\ \xi_{3,3,4},\ \eta_{3,3,5},\ \eta_{3,4,1},\ \eta_{3,5,1},\ \eta_{4,2,1},$$
$$\eta_{4,3,1},\ \eta_{4,4,1},\ \eta_{4,4,2},\ \eta_{4,4,3},\ \xi_{4,4,3},\ \eta_{4,4,4},\ \xi_{4,4,4},\ \eta_{5,1,1},\ \eta_{5,3,1},\ \eta_{5,5,1},\ \eta_{5,5,3},$$

where $|\mathcal{G}| = 195$ and the largest relation found is of degree 6.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 40$:

$$H^*(S_8, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4, x_7, x_8] / \langle x_3^2, x_7^2 \rangle.$$

The number of generators is 4 and $|\mathcal{G}| = 2$.

The Ext-algebra computation for $n = 30$ produces the following 20 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{1,4,1},\ \eta_{2,1,1},\ \eta_{2,1,2},\ \eta_{2,3,1},\ \eta_{2,3,2},\ \eta_{2,5,1},\ \eta_{2,5,2},\ \eta_{3,2,1},$$

$$\eta_{3,2,2},\ \eta_{3,4,1},\ \eta_{4,1,1},\ \eta_{4,3,1},\ \eta_{4,4,3},\ \eta_{4,4,8},\ \eta_{4,5,2},\ \eta_{5,2,1},\ \eta_{5,2,2},\ \eta_{5,4,2},$$

where $|\mathcal{G}| = 97$ and the largest relation found is of degree 30.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*\left(S_8, \mathbb{F}_5\right) \cong \mathbb{F}_5\left[x_7, x_8\right]/\langle x_7^2\rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 10 generators:

$$\eta_{1,2,4},\ \eta_{1,4,1},\ \eta_{2,1,4},\ \eta_{2,3,1},\ \eta_{3,2,1},\ \eta_{3,4,1},\ \eta_{3,4,4},\ \eta_{4,1,1},\ \eta_{4,3,1},\ \eta_{4,3,4},$$

where $|\mathcal{G}| = 14$ and the largest relation found is of degree 5.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*\left(S_8, \mathbb{F}_7\right) \cong \mathbb{F}_7\left[x_{11}, x_{12}\right]/\langle x_{11}^2\rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 16 generators:

$$\eta_{1,2,1},\ \eta_{1,3,6},\ \eta_{2,1,1},\ \eta_{2,4,6},\ \eta_{2,6,1},\ \eta_{3,1,6},\ \eta_{3,4,1},\ \eta_{4,2,6},$$

$$\eta_{4,3,1}, \eta_{4,5,1},\ \eta_{5,4,1},\ \eta_{5,6,1},\ \eta_{5,6,6},\ \eta_{6,2,1},\ \eta_{6,5,1},\ \eta_{6,5,6},$$

where $|\mathcal{G}| = 26$ and the largest relation found is of degree 7.

5.4.6 S_9

The order of S_9 is $2^7 \cdot 3^4 \cdot 5 \cdot 7 = 362880$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 12$:

$$H^*(S_9, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, x_3, y_3, x_4, x_5, x_6, x_7] / \langle \mathcal{G} \rangle.$$

The number of generators is 8 and $|\mathcal{G}| = 24$.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 20$:

$$H^*(S_9, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4, x_7, x_8, x_{10}, x_{11}, y_{11}, x_{12}, x_{15}, x_{16}] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is the set:

$$x_3^2, x_3x_{10}, x_3y_{11}, x_4x_{10}, x_4y_{11}, x_7^2, x_7x_{10}, x_7y_{11}, x_8x_{10}, x_8y_{11}, x_{10}^2,$$
$$x_3x_8^2 + x_3x_{16} + 2 \cdot x_3x_4x_{12}, x_3x_4^2x_7 + x_3x_{15} + 2 \cdot x_3x_4x_{11},$$
$$x_4x_8^2 + x_4x_{16} + 2 \cdot x_4^2x_{12}, x_4^3x_7 + x_4x_{15} + 2 \cdot x_3x_4x_{12} + 2 \cdot x_4^2x_4x_{11} + x_3x_4^2x_8.$$

The number of generators is 10 and $|\mathcal{G}| = 15$.

The Ext-algebra computation for $n = 6$ produces 72 generators where the largest generator found is of degree 5 and $|\mathcal{G}| = 423$.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(S_9, \mathbb{F}_5) \cong \mathbb{F}_5[x_7, x_8] / \langle x_7^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 10 generators:

$$\eta_{1,2,4},\ \eta_{1,3,1},\ \eta_{2,1,4},\ \eta_{2,4,1},\ \eta_{3,1,1},\ \eta_{3,4,1},\ \eta_{3,4,4},\ \eta_{4,2,1},\ \eta_{4,3,1},\ \eta_{4,3,4},$$

where $|\mathcal{G}| = 14$ and the largest relation found is of degree 5.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(S_9, \mathbb{F}_7) \cong \mathbb{F}_7[x_{11}, x_{12}] / \langle x_{11}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 16 generators:

$$\eta_{1,3,1},\ \eta_{1,5,1},\ \eta_{1,6,6},\ \eta_{2,3,1},\ \eta_{2,3,6},\ \eta_{2,6,1},\ \eta_{3,1,1},\ \eta_{3,2,1},$$

$$\eta_{3,2,6},\ \eta_{4,5,6},\ \eta_{4,6,1},\ \eta_{5,1,1},\ \eta_{5,4,6},\ \eta_{6,1,6},\ \eta_{6,2,1},\ \eta_{6,4,1},$$

where $|\mathcal{G}| = 31$ and the largest relation found is of degree 7.

5.4.7 S_{10}

The order of S_{10} is $2^8 \cdot 3^4 \cdot 5^2 \cdot 7 = 3628800$.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 20$:

$$H^*(S_{10}, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4, x_7, x_8, x_{10}, x_{11}, y_{11}, x_{12}, x_{15}, x_{16}] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is the set:

$$\begin{aligned}
x_3^2, x_3x_{10}, x_3y_{11} + x_3x_{11}, x_4x_{10}, x_7^2, x_7x_{10}, x_3x_4x_{11} + x_3x_{15},\\
x_8x_{10}, x_4^2y_{11} + 2 \cdot x_4^2x_{11} + x_4x_{15} + x_3x_4x_{12}, x_{10}^2, x_3x_8^2 + 2 \cdot x_3x_{16} + 2 \cdot x_3x_4x_{12},\\
x_3x_4^3 + x_3x_4x_8 + 2 \cdot x_4y_{11} + 2 \cdot x_4x_{11}, x_3x_4^2x_4x_7 + x_3x_7x_8 + x_7y_{11} + x_7x_{11},\\
x_3x_4^2x_8 + x_3x_{16} + 2 \cdot x_8x_{11} + x_3x_4x_{12} + 2 \cdot x_8y_{11}, x_4x_8^2 + 2 \cdot x_4x_{16} + 2 \cdot x_4^2x_{12}.
\end{aligned}$$

The number of generators is 10 and $|\mathcal{G}| = 15$.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 30$:

$$H^*(S_{10}, \mathbb{F}_5) \cong \mathbb{F}_5[x_7, x_8, x_{15}, x_{16}] / \langle x_7^2, x_{15}^2 \rangle.$$

The number of generators is 4 and $|\mathcal{G}| = 2$.

The Ext-algebra computation for $n = 20$ produces 151 generators where the largest degree of generator found is 16 and $|\mathcal{G}| = 1793$.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(S_{10}, \mathbb{F}_7) \cong \mathbb{F}_7[x_{11}, x_{12}] / \langle x_{11}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 16 generators:

$$\eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{1,6,6},\ \eta_{2,1,1},\ \eta_{2,4,6},\ \eta_{3,1,1},\ \eta_{3,5,1},\ \eta_{3,5,6},$$
$$\eta_{4,2,6},\ \eta_{4,6,1},\ \eta_{5,3,1},\ \eta_{5,3,6},\ \eta_{5,6,1},\ \eta_{6,1,6},\ \eta_{6,4,1},\ \eta_{6,5,1},$$

where $|\mathcal{G}| = 27$ and the largest relation found is of degree 7.

5.5 Sporadic Simple Groups

5.5.1 M_{11}

The order of the Mathieu group M_{11} is $2^4 \cdot 3^2 \cdot 5 \cdot 11 = 7920$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 100$:

$$H^*(M_{11}, \mathbb{F}_2) \cong \mathbb{F}_2[x_3, x_4, x_5] / \langle x_3^2 x_4 + x_5^2 \rangle.$$

The number of generators is 3 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 8 generators:

$$\eta_{1,1,4},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,4},\ \eta_{3,1,1},\ \eta_{3,3,1},$$

where $|\mathcal{G}| = 27$ and the largest relation found is of degree 30.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 100$:

$$H^*(M_{11}, \mathbb{F}_3) \cong \mathbb{F}_3[x_7, x_8, x_{10}, x_{11}, y_{11}, x_{12}, x_{15}, x_{16}] / \langle \mathcal{G} \rangle.$$

The number of generators is 8 and $|\mathcal{G}| = 20$.

The Ext-algebra computation for $n = 30$ produces the following 48 generators:

$$\eta_{1,1,2},\ \eta_{1,2,1},\ \eta_{1,2,6},\ \eta_{1,3,5},\ \eta_{1,3,6},\ \eta_{1,3,9},\ \eta_{1,3,10},\ \eta_{1,4,4},\ \eta_{1,4,9},\ \eta_{1,6,2},\ \eta_{1,7,1},\ \eta_{1,7,6},$$

$$\eta_{2,3,1},\ \eta_{2,3,6},\ \eta_{2,4,1},\ \eta_{2,4,2},\ \eta_{2,4,5},\ \eta_{2,4,6},\ \eta_{2,5,3},\ \eta_{2,5,8},\ \eta_{2,7,2},\ \eta_{3,1,1},\ \eta_{3,1,6},\ \eta_{3,3,2},$$

$$\eta_{3,4,2},\ \eta_{3,5,1},\ \eta_{3,5,4},\ \eta_{3,6,1},\ \eta_{4,1,1},\ \eta_{4,3,2},\ \eta_{4,6,4},\ \eta_{5,1,1},\ \eta_{5,1,4},\ \eta_{5,2,3},\ \eta_{5,2,8},\ \eta_{6,1,2},$$

$$\eta_{6,2,1},\ \eta_{6,2,2},\ \eta_{6,2,5},\ \eta_{6,2,6},\ \eta_{6,3,4},\ \eta_{6,3,9},\ \eta_{7,2,2},\ \eta_{7,3,1},\ \eta_{7,3,6},\ \eta_{7,7,2},\ \eta_{7,7,11},\ \eta 7, 7, 16,$$

where $|\mathcal{G}| = 428$ and the largest relation found is of degree 30.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(M_{11}, \mathbb{F}_5) \cong \mathbb{F}_5[x_7, x_8] / \langle x_7^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 8 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{2,3,1},\ \eta_{2,3,2},\ \eta_{3,4,1},\ \eta_{3,4,2},\ \eta_{4,1,1},\ \eta_{4,1,2},$$

where $\mathcal{G}$ is the set:

$$\eta_{2,3,1}\eta_{1,2,1},\ \eta_{3,4,1}\eta_{2,3,1},\ \eta_{4,1,1}\eta_{3,4,1},\ \eta_{1,2,1}\eta_{4,1,1},\ \eta_{2,3,1}\eta_{1,2,2} + Z(5)^2\cdot$$
$$\eta_{2,3,2}\eta_{1,2,1},\ \eta_{3,4,1}\eta_{2,3,2} + Z(5)^2 \cdot \eta_{3,4,2}\eta_{2,3,1},\ \eta_{4,1,1}\eta_{3,4,2} + Z(5)^2\cdot$$
$$\eta_{4,1,2}\eta_{3,4,1},\ \eta_{1,2,1}\eta_{4,1,2} + Z(5)^2 \cdot \eta_{1,2,2}\eta_{4,1,1}.$$

Characteristic 11: For the splitting field $\mathbb{F}_{11}$ with degree of computation $n = 100$:

$$H^*(M_{11}, \mathbb{F}_{11}) \cong \mathbb{F}_{11}[x_9, x_{10}] / \langle x_9^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 14 generators:

$$\eta_{1,1,10},\ \eta_{1,2,1},\ \eta_{1,4,1},\ \eta_{1,5,3},\ \eta_{2,3,4},\ \eta_{2,5,1},\ \eta_{3,1,1},$$

$$\eta_{3,4,3},\ \eta_{4,1,1},\ \eta_{4,2,3},\ \eta_{5,1,3},\ \eta_{5,3,1},\ \eta_{5,5,1},\ \eta_{5,5,10},$$

where $|\mathcal{G}| = 34$ and the largest relation found is of degree 13:

$$\eta_{2,5,1}\eta_{1,2,1},\ \eta_{5,3,1}\eta_{2,5,1},\ \eta_{1,2,1}\eta_{3,1,1},\ \eta_{1,4,1}\eta_{4,1,1},\ \eta_{3,1,1}\eta_{5,3,1},\ \eta_{1,2,1}\eta_{4,1,1}\eta_{1,4,1},$$
$$\eta_{5,5,1}\eta_{5,5,1}\eta_{2,5,1},\ \eta_{4,1,1}\eta_{1,4,1}\eta_{3,1,1},\ \eta_{5,3,1}\eta_{5,5,1}\eta_{5,5,1},\ \eta_{5,1,3}\eta_{2,5,1},\ \eta_{1,5,3}\eta_{3,1,1},\ \eta_{5,3,1}\eta_{1,5,3},$$
$$\eta_{2,5,1}\eta_{4,2,3} + Z(11)^3 \cdot \eta_{1,5,3}\eta_{4,1,1},\ \eta_{1,2,1}\eta_{5,1,3},$$
$$\eta_{1,4,1}\eta_{5,1,3} + Z(11)^2 \cdot \eta_{3,4,3}\eta_{5,3,1},\ \eta_{5,3,1}\eta_{5,5,1}\eta_{1,5,3} + Z(11)^7 \cdot \eta_{2,3,4}\eta_{1,2,1},$$
$$\eta_{5,5,1}\eta_{5,5,1}\eta_{1,5,3} + Z(11)^7 \cdot \eta_{1,5,3}\eta_{4,1,1}\eta_{1,4,1},$$
$$\eta_{3,1,1}\eta_{2,3,4} + Z(11) \cdot \eta_{5,1,3}\eta_{5,5,1}\eta_{2,5,1},\ \eta_{1,2,1}\eta_{4,1,1}\eta_{3,4,3} + Z(11)^2 \cdot \eta_{4,2,3}\eta_{1,4,1}\eta_{3,1,1},$$
$$\eta_{4,1,1}\eta_{3,4,3}\eta_{5,3,1} + Z(11)^6 \cdot \eta_{5,1,3}\eta_{5,5,1}\eta_{5,5,1},\ \eta_{5,1,3}\eta_{1,5,3},$$
$$\eta_{1,5,3}\eta_{5,1,3} + Z(11)^9 \cdot \eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1},$$
$$\eta_{5,1,3}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1},$$
$$\eta_{4,1,1}\eta_{3,4,3}\eta_{2,3,4}\eta_{1,2,1} + Z(11)^6 \cdot \eta_{5,1,3}\eta_{5,5,1}\eta_{1,5,3}\eta_{4,1,1}\eta_{1,4,1},$$
$$\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1}\eta_{5,5,1},$$
$$\eta_{1,5,3}\eta_{4,1,1}\eta_{3,4,3}\eta_{2,3,4} + Z(11) \cdot \eta_{5,5,10}\eta_{2,5,1},$$
$$\eta_{5,1,3}\eta_{5,5,1}\eta_{1,5,3}\eta_{4,1,1}\eta_{3,4,3} + Z(11)^5 \cdot \eta_{1,1,10}\eta_{3,1,1},$$
$$\eta_{1,2,1}\eta_{1,1,10} + Z(11) \cdot \eta_{4,2,3}\eta_{3,4,3}\eta_{2,3,4}\eta_{1,2,1},\ \eta_{1,4,1}\eta_{1,1,10} + Z(11) \cdot \eta_{3,4,3}\eta_{2,3,4}\eta_{4,2,3}\eta_{1,4,1},$$
$$\eta_{4,1,1}\eta_{3,4,3}\eta_{2,3,4}\eta_{4,2,3} + Z(11)^9 \cdot \eta_{1,1,10}\eta_{4,1,1},\ \eta_{5,3,1}\eta_{5,5,10} + Z(11) \cdot \eta_{2,3,4}\eta_{4,2,3}\eta_{3,4,3}\eta_{5,3,1},$$
$$\eta_{5,5,1}\eta_{5,5,10} + Z(11)^5 \cdot \eta_{5,5,10}\eta_{5,5,1},\ \eta_{1,5,3}\eta_{1,1,10} + Z(11)^5 \cdot \eta_{5,5,10}\eta_{1,5,3},$$
$$\eta_{5,1,3}\eta_{5,5,10} + Z(11)^5 \cdot \eta_{1,1,10}\eta_{5,1,3}.$$

5.5.2 M_{12}

The order of the Mathieu group M_{12} is $2^6 \cdot 3^3 \cdot 5 \cdot 11 = 95040$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 12$:

$$H^*(M_{12}, \mathbb{F}_2) \cong \mathbb{F}_2[x_2, x_3, y_3, z_3, x_4, x_5, x_6, x_7]/\langle\mathcal{G}\rangle.$$

where $\mathcal{G}$ is the set

$$\begin{aligned} x_2y_3,\ x_3y_3,\ x_3x_5+x_2x_3z_3,\ y_3z_3+y_3y_3,\ z_3^2+y_3^2+x_2^3+x_3^2+x_3z_3, \\ z_3x_6+x_2^2x_5+x_2x_3x_4+x_2z_3x_4+x_3x_6+x_3^3+y_3x_6, \\ z_3x_5+x_2^4+x_2x_3^2+x_2x_3z_3+y_3x_5,\ z_3x_7+x_2^2x_6+x_2^3x_4+x_2^5+x_3z_3x_4+y_3x_7, \\ x_2^2z_3+x_2x_5,\ x_5x_5+x_2^5+x_2^2x_3^2+x_2^2x_3z_3+y_3x_7, \\ x_5x_7+x_2^3x_6+x_2^4x_4+x_2^5x_2+x_2x_3z_3x_4+y_3^2x_6+y_3x_4x_5, \\ x_2x_4x_5+x_2^2z_3x_4,\ x_3^2z_3+x_2x_7+x_3x_6, \\ x_3^2x_4+x_2^2x_6+dotx_2^3x_4+x_2^5+x_2^2x_3^2+x_2^2x_3z_3+x_3x_7. \end{aligned}$$

The number of generators is 8 and $|\mathcal{G}|=14$.

The Ext-algebra computation for $n=12$ produces the following 27 generators:

$$\begin{gathered} \eta_{1,1,4},\ \eta_{1,1,6},\ \eta_{1,1,7},\ \eta_{1,2,1},\ \xi_{1,2,1},\ \eta_{1,3,1},\ \eta_{1,3,2},\ \eta_{1,3,3},\ \eta_{2,1,1}, \\ \xi_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,3},\ \eta_{2,2,4},\ \eta_{2,2,6},\ \xi_{2,2,6},\ \eta_{2,3,1},\ \eta_{3,1,1},\ \eta_{3,1,2}, \\ \eta_{3,1,3},\ \eta_{3,2,1},\ \eta_{3,3,1},\ \eta_{3,3,3},\ \eta_{3,3,4},\ \eta_{3,3,6},\ \xi_{3,3,6},\ \eta_{3,3,7},\ \eta_{3,3,8}, \end{gathered}$$

where $|\mathcal{G}|=251$ and the largest relation found is of degree 12.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n=30$:

$$H^*(M_{12},\mathbb{F}_3)\cong \\ \mathbb{F}_3[x_3,x_4,y_4,x_5,x_9,x_{10},y_{10},z_{10},x_{11},y_{11},z_{11},x_{12},x_{15},y_{15},x_{16},y_{16}]/\langle\mathcal{G}\rangle.$$

The number of generators is 16 and $|\mathcal{G}|=105$.

The Ext-algebra computation for $n=30$ produces 58 generators where the largest generator found is of degree 16.

$|\mathcal{G}|=449$ and the largest relation found is of degree 30.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(M_{12}, \mathbb{F}_5) \cong \mathbb{F}_5[x_7, x_8] / \langle x_7^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 10 generators:

$$\eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{1,3,4},\ \eta_{2,1,1},\ \eta_{2,4,4},\ \eta_{3,1,1},\ \eta_{3,1,4},\ \eta_{3,4,1},\ \eta_{4,2,4},\ \eta_{4,3,1},$$

where $|\mathcal{G}| = 15$ and the largest relation found is of degree 5.

Characteristic 11: For the splitting field $\mathbb{F}_{11}$ with degree of computation $n = 100$:

$$H^*(M_{12}, \mathbb{F}_{11}) \cong \mathbb{F}_{11}[x_9, x_{10}] / \langle x_9^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 14 generators:

$$\eta_{1,1,10},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,10},\ \eta_{2,4,1},\ \eta_{3,1,1},$$
$$\eta_{3,3,10},\ \eta_{3,5,1},\ \eta_{4,2,1},\ \eta_{4,4,10},\ \eta_{5,3,1},\ \eta_{5,5,1},\ \eta_{5,5,10},$$

where $|\mathcal{G}| = 29$ and the largest relation found is of degree 11.

5.5.3 J_1

The order of the Janko group J_1 is $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 = 175560$.

Characteristic 2: For the splitting field $\mathbb{F}_4$ with degree of computation $n = 30$:

$$H^*(J_1, \mathbb{F}_4) \cong \mathbb{F}_4[x_3, x_4, x_5, x_6, x_7] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is the following set:

$$x_5^2 + Z(4) \cdot x_3x_7 + Z(4)^2 \cdot x_4x_6,$$
$$x_3^4 + Z(4) \cdot x_3^2x_6 + Z(4) \cdot x_5x_7 + Z(4) \cdot x_3x_4x_5 + Z(4)^2 \cdot x_6^2 + x_4^3.$$

The number of generators is 5 and $|\mathcal{G}| = 2$.

The Ext-algebra computation for $n = 30$ produces the following 35 generators:

$$\eta_{1,1,3},\ \eta_{1,1,4},\ \eta_{1,1,5},\ \eta_{1,1,6},\ \eta_{1,1,7},\ \eta_{1,3,1},\ \eta_{1,3,2},\ \eta_{1,4,1},\ \eta_{1,4,2},\ \eta_{1,5,1},\ \eta_{2,2,7},\ \eta_{2,3,1},$$
$$\eta_{2,4,1},\ \eta_{2,5,2},\ \eta_{3,1,1},\ \eta_{3,1,2},\ \eta_{3,2,1},\ \eta_{3,3,3},\ \eta_{3,3,4},\ \eta_{3,3,5},\ \eta_{3,3,6},\ \eta_{3,3,7},\ \eta_{4,1,1},\ \eta_{4,1,2},$$
$$\eta_{4,2,1},\ \eta_{4,4,3},\ \eta_{4,4,4},\ \eta_{4,4,5},\ \eta_{4,4,6},\ \eta_{4,4,7},\ \eta_{5,1,1},\ \eta_{5,2,2},\ \eta_{5,5,1},\ \eta_{5,5,4},\ \eta_{5,5,7},$$

where $|\mathcal{G}| = 277$ and the largest relation found is of degree 30.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 100$:

$$H^*(J_1, \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4]/\langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 4 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{2,1,1},\ \eta_{2,1,2},$$

where $\mathcal{G}$ is the set:

$$\eta_{2,1,1}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,1},\ \eta_{2,1,1}\eta_{1,2,2} + 2 \cdot \eta_{2,1,2}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,2} + 2 \cdot \eta_{1,2,2}\eta_{2,1,1}.$$

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(J_1, \mathbb{F}_5) \cong \mathbb{F}_5[x_3, x_4]/\langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 4 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{2,1,1},\ \eta_{2,1,2},$$

with $\mathcal{G}$ the set:

$$\eta_{2,1,1}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,1},\ \eta_{2,1,1}\eta_{1,2,2} + Z(5)^2 \cdot \eta_{2,1,2}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,2} + Z(5)^2 \cdot \eta_{1,2,2}\eta_{2,1,1}.$$

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(J_1, \mathbb{F}_7) \cong \mathbb{F}_7[x_{11}, x_{12}]/\langle x_{11}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 16 generators:

$$\eta_{1,2,1},\ \eta_{1,4,6},\ \eta_{1,5,1},\ \eta_{2,1,1},\ \eta_{2,3,1},\ \eta_{2,3,6},\ \eta_{3,2,1},\ \eta_{3,2,6},$$

$$\eta_{3,4,1},\ \eta_{4,1,6},\ \eta_{4,3,1},\ \eta_{4,6,1},\ \eta_{5,1,1},\ \eta_{5,6,6},\ \eta_{6,4,1},\ \eta_{6,5,6},$$

where $|\mathcal{G}| = 31$ and the largest relation found is of degree 7.

Characteristic 11: For the splitting field $\mathbb{F}_{11}$ with degree of computation $n = 100$:

$$H^*(J_1, \mathbb{F}_{11}) \cong \mathbb{F}_{11}[x_{19}, x_{20}]/\langle x_{19}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 28 generators:

$$\eta_{1,2,1},\ \eta_{1,7,1},\ \eta_{1,7,10},\ \eta_{2,1,1},\ \eta_{2,6,1},\ \eta_{2,9,10},\ \eta_{3,6,10},\ \eta_{3,9,1},\ \eta_{3,10,1},\ \eta_{4,5,10},$$

$$\eta_{4,10,1},\ \eta_{5,4,10},\ \eta_{5,8,1},\ \eta_{6,2,1},\ \eta_{6,3,10},\ \eta_{6,8,1},\ \eta_{7,1,1},\ \eta_{7,1,10},\ \eta_{7,9,1},$$

$$\eta_{8,5,1},\ \eta_{8,6,1},\ \eta_{8,10,10},\ \eta_{9,2,10},\ \eta_{9,3,1},\ \eta_{9,7,1},\ \eta_{10,3,1},\ \eta_{10,4,1},\ \eta_{10,8,10},$$

where $|\mathcal{G}| = 64$ and the largest relation found is of degree 11.

Characteristic 19: For the splitting field $\mathbb{F}_{19}$ with degree of computation $n = 100$:

$$H^*(J_1, \mathbb{F}_{19}) \cong \mathbb{F}_{19}[x_{11}, x_{12}]/\langle x_{11}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 17 generators:

$$\eta_{1,1,3},\ \eta_{1,1,12},\ \eta_{1,2,1},\ \eta_{1,6,1},\ \eta_{2,1,1},\ \eta_{2,2,12},\ \eta_{2,3,1},\ \eta_{3,2,1},\ \eta_{3,3,12},$$

$$\eta_{3,5,1},\ \eta_{4,5,1},\ \eta_{4,6,6},\ \eta_{5,3,1},\ \eta_{5,4,1},\ \eta_{5,5,12},\ \eta_{6,1,1},\ \eta_{6,4,6},$$

where $|\mathcal{G}| = 45$ and the largest relation found is of degree 15.

5.5.4 M_{22}

The order of the Mathieu group M_{22} is $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 443520$.

Characteristic 2: For the splitting field $\mathbb{F}_4$ with degree of computation $n = 15$:

$$H^*(M_{22}, \mathbb{F}_4) \cong \mathbb{F}_4[x_2, x_3, x_5, y_5, x_6, y_6, x_7, x_8, y_8, x_9, y_9, x_{10}, x_{11}, x_{12}, y_{12}] / \langle \mathcal{G} \rangle.$$

The number of generators is 15 and $|\mathcal{G}| = 32$.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 50$:

$$H^*(M_{22}, \mathbb{F}_3) \cong \mathbb{F}_3[x_2, x_3, x_7, y_7, x_8, y_8, x_{11}, x_{12}] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is the set:

$$\begin{aligned} &x_2^2, x_2x_3, x_2x_7, x_2y_7, x_2x_{11}, x_3^2, x_3x_7 + Z(3) \cdot x_2x_8 + x_2y_8, x_3y_7 + Z(3) \cdot x_2y_8, \\ &x_3x_{11} + x_2x_{12}, x_7^2, x_7y_7 + x_2x_{12}, x_7x_{11} + Z(3) \cdot x_2x_8y_8 + x_2y_8^2, y_7^2, \\ &y_7y_8 + x_3x_{12} + x_7y_8 + Z(3) \cdot y_7x_8, y_7x_{11} + x_2x_8y_8 + Z(3) \cdot x_2y_8^2, \\ &x_8x_{11} + x_7x_{12} + y_7x_{12}, y_8x_{11} + Z(3) \cdot x_3x_8y_8 + x_3y_8^2 + y_7x_{12}, x_{11}^2, \\ &x_{11}x_{12} + x_3y_8x_{12} + x_7x_8y_8, x_{12}^2 + Z(3) \cdot x_8^2y_8 + x_8y_8^2. \end{aligned}$$

The number of generators is 8 and $|\mathcal{G}| = 20$.

The Ext-algebra computation for $n = 30$ produces the following 42 generators:

$$\begin{gathered} \eta_{1,1,3},\ \eta_{1,1,8},\ \eta_{1,2,1},\ \eta_{1,4,1},\ \eta_{2,1,1},\ \eta_{2,2,3},\ \eta_{2,2,8},\ \eta_{2,3,1},\ \eta_{2,3,6},\ \eta_{2,4,3},\ \eta_{2,4,4}, \\ \eta_{2,4,8},\ \eta_{2,5,1},\ \eta_{2,5,6},\ \eta_{3,2,1},\ \eta_{3,2,6},\ \eta_{3,3,3},\ \eta_{3,3,8},\ \eta_{3,4,1},\ \eta_{3,4,6},\ \eta_{3,5,3},\ \eta_{3,5,4}, \\ \eta_{3,5,8},\ \eta_{4,1,1},\ \eta_{4,2,3},\ \eta_{4,2,4},\ \eta_{4,2,8},\ \eta_{4,3,1},\ \eta_{4,3,6},\ \eta_{4,4,3},\ \eta_{4,4,8},\ \eta_{4,5,1}, \\ \eta_{4,5,6},\ \eta_{5,2,1},\ \eta_{5,2,6},\ \eta_{5,3,3},\ \eta_{5,3,4},\ \eta_{5,3,8},\ \eta_{5,4,1},\ \eta_{5,4,6},\ \eta_{5,5,3},\ \eta_{5,5,8}, \end{gathered}$$

where $|\mathcal{G}| = 291$ and the largest relation found is of degree 16.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(M_{22}, \mathbb{F}_5) \cong \mathbb{F}_5 [x_7, x_8] / \langle x_7^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 10 generators:

$$\eta_{1,2,1},\ \eta_{1,2,4},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,1,4},\ \eta_{2,4,1},\ \eta_{3,1,1},\ \eta_{3,4,4},\ \eta_{4,2,1},\ \eta_{4,3,4},$$

where $|\mathcal{G}| = 16$ and the largest relation found is of degree 5.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(M_{22}, \mathbb{F}_7) \cong \mathbb{F}_7 [x_5, x_6] / \langle x_5^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 8 generators:

$$\eta_{1,1,6},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,6},\ \eta_{3,1,1},\ \eta_{3,3,6},$$

where $|\mathcal{G}| = 14$ and the largest relation found is of degree 7.

Characteristic 11: For the splitting field $\mathbb{F}_{11}$ with degree of computation $n = 100$:

$$H^*(M_{22}, \mathbb{F}_{11}) \cong \mathbb{F}_{11} [x_9, x_{10}] / \langle x_9^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 13 generators:

$$\eta_{1,1,5},\ \eta_{1,1,10},\ \eta_{1,2,1},\ \eta_{1,4,1},\ \eta_{2,3,1},\ \eta_{2,3,2},\ \eta_{3,1,1},\ \eta_{3,5,4},\ \eta_{4,1,1},\ \eta_{4,4,10},\ \eta_{4,5,1},\ \eta_{5,2,4},\ \eta_{5,4,1},$$

where $|\mathcal{G}| = 30$ and the largest relation found is of degree 15.

5.5.5 J_2

The order of the Janko group J_2 is $2^7 \cdot 3^3 \cdot 5^2 \cdot 7 = 604800$.

Characteristic 2: For the splitting field $\mathbb{F}_4$ with degree of computation $n = 10$:

$$H^*(J_2, \mathbb{F}_4) \cong \mathbb{F}_4 [x_2, x_3, y_3, x_5, x_6, x_7, x_8, y_8, z_8, x_9, y_9] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is the set:

$$\begin{aligned} x_3x_5 + Z(2^2)^2 \cdot x_2x_3^2, x_3x_7 + Z(2^2) \cdot x_2^2x_3^2 + Z(2^2) \cdot x_2^2x_3y_3, \\ y_3^2 + Z(2^2) \cdot x_2^3 + Z(2^2) \cdot x_3y_3, y_3x_5 + Z(2^2)^2 \cdot x_2x_3y_3, \\ y_3x_7 + Z(2^2)^2 \cdot x_2^5 + x_2^2x_3y_3, x_2^2x_3 + Z(2^2) \cdot x_2x_5, \\ x_5^2 + Z(2^2) \cdot x_2^2x_3^2, x_2^3y_3 + Z(2^2)^2 \cdot x_2x_7 + Z(2^2) \cdot x_2^2x_5. \end{aligned}$$

The number of generators is 11 and $|\mathcal{G}| = 8$.

Characteristic 3: For the splitting field $\mathbb{F}_9$ with degree of computation $n = 30$:

$$H^*(J_2, \mathbb{F}_9) \cong \mathbb{F}_9 [x_3, x_4, y_4, x_5, x_9] / \langle \mathcal{G} \rangle.$$

where $|\mathcal{G}| = 22$ and the largest relation found is of degree 25.

The Ext-algebra computation for $n = 20$ produces generators where the largest generator found is of degree 12 and $|\mathcal{G}| = 473$ and the largest relation found is of degree 20.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 40$:

$$H^*(J_2, \mathbb{F}_5) \cong \mathbb{F}_5 [x_3, x_4, x_{11}, x_{12}] / \langle x_3^2, x_{11}^2 \rangle.$$

The number of generators is 4 and $|\mathcal{G}| = 2$.

The Ext-algebra computation for $n = 24$ produces the following 33 generators:

$$\begin{aligned} &\eta_{1,1,4},\ \eta_{1,1,12},\ \eta_{1,2,1},\ \eta_{1,6,8},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,4},\ \eta_{2,2,8},\ \eta_{2,2,12},\ \eta_{2,4,1}, \eta_{2,5,1}, \\ &\eta_{2,6,1},\ \eta_{3,3,12},\ \eta_{3,4,1},\ \eta_{3,6,1},\ \eta_{4,2,1},\ \eta_{4,3,1},\ \eta_{4,4,1},\ \eta_{4,4,4},\ \eta_{4,4,12},\ \eta_{4,5,1},\ \eta_{4,5,8}, \\ &\eta_{5,2,1},\ \eta_{5,4,1},\ \eta_{5,4,8}, \eta_{5,5,1},\ \eta_{5,5,4},\ \eta_{5,5,12},\ \eta_{6,1,8},\ \eta_{6,2,1},\ \eta_{6,3,1},\ \eta_{6,6,4},\ \eta_{6,6,12}, \end{aligned}$$

where $|\mathcal{G}| = 275$ and the largest relation found is of degree 20.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(J_2, \mathbb{F}_7) \cong \mathbb{F}_7 [x_{11}, x_{12}] / \langle x_{11}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 16 generators:

$$\eta_{1,4,6},\ \eta_{1,5,1},\ \eta_{2,5,1},\ \eta_{2,6,1},\ \eta_{2,6,6}, \eta_{3,4,1},\ \eta_{3,5,6},\ \eta_{3,6,1},$$

$$\eta_{4,1,6},\ \eta_{4,3,1}, \eta_{5,1,1},\ \eta_{5,2,1},\ \eta_{5,3,6},\ \eta_{6,2,1},\ \eta_{6,2,6},\ \eta_{6,3,1},$$

where $|\mathcal{G}| = 27$ and the largest relation found is of degree 7.

5.5.6 M_{23}

The order of the Mathieu group M_{23} is $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 10200960$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 14$:

$$H^*(M_{23}, \mathbb{F}_2) \cong \mathbb{F}_2 [x_6, x_7, y_7, x_8, y_8, x_9, x_{10}, x_{11}, y_{11}, z_{11}, x_{12}, y_{12}, x_{13}] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is the set:

$$x_6y_7 + x_6x_7, x_7y_7 + x_7^2, y_7^2 + x_7^2.$$

The number of generators is 13 and $|\mathcal{G}| = 3$.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 40$:

$$H^*(M_{23}, \mathbb{F}_3) \cong \mathbb{F}_3 [x_7, x_8, x_{10}, x_{11}, y_{11}, x_{12}, x_{15}, x_{16}] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is the set:

$$x_7^2, x_7x_{10}, x_7y_{11} + 2 \cdot x_7x_{11}, x_8x_{10} + 2 \cdot x_7x_{11}, x_8y_{11} + x_7x_{12} + 2 \cdot x_8x_{11},$$

$$x_{10}^2, x_{10}x_{11}, x_{10}y_{11}, x_{10}x_{12} + 2 \cdot x_7x_{15}, x_{10}x_{15}, x_{11}^2, x_{11}y_{11} + 2 \cdot x_7x_{15},$$

$$x_{11}x_{12} + 2 \cdot x_7x_{16} + 2 \cdot x_8x_{15}, x_{11}x_{15} + 2 \cdot x_{10}x_{16}, y_{11}^2, y_{11}x_{12} + 2 \cdot x_8x_{15},$$

$$y_{11}x_{15}, x_{12}^2 + 2 \cdot x_8x_{16}, x_{12}x_{15} + 2 \cdot y_{11}x_{16}, x_{15}^2.$$

The number of generators is 8 and $|\mathcal{G}| = 20$.

The Ext-algebra computation for $n = 30$ produces the following 36 generators:

$$\eta_{1,1,3},\ \eta_{1,1,4},\ \eta_{1,5,2},\ \eta_{1,6,1},\ \eta_{1,6,2},\ \eta_{1,7,1},\ \eta_{1,7,2},\ \eta_{2,1,3},\ \eta_{2,1,4},\ \eta_{2,2,3},\ \eta_{2,2,4},\ \eta_{2,3,1},$$
$$\eta_{2,3,2},\ \eta_{2,4,4},\ \eta_{2,5,1},\ \eta_{3,1,1},\ \eta_{3,1,2},\ \eta_{3,5,1},\ \eta_{4,1,1},\ \eta_{4,2,2},\ \eta_{4,3,1},\ \eta_{4,4,2},\ \eta_{4,5,1},\ \eta_{4,5,6},$$
$$\eta_{5,1,4},\ \eta_{5,4,5},\ \eta_{5,4,10},\ \eta_{5,5,2},\ \eta_{5,6,1},\ \eta_{5,7,3},\ \eta_{6,2,1},\ \eta_{6,2,2},\ \eta_{6,4,1},\ \eta_{7,2,1},\ \eta_{7,2,2},\ \eta_{7,4,3},$$

where $|\mathcal{G}| = 236$ and the largest relation found is of degree 21.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*(M_{23}, \mathbb{F}_5) \cong \mathbb{F}_5 [x_7, x_8] / \langle x_7^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 8 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{2,3,1},\ \eta_{2,3,2},\ \eta_{3,4,1},\ \eta_{3,4,2},\ \eta_{4,1,1},\ \eta_{4,1,2},$$

where $|\mathcal{G}| = 8$ and the largest relation found is of degree 3.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(M_{23}, \mathbb{F}_7) \cong \mathbb{F}_7 [x_5, x_6] / \langle x_5^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 8 generators:

$$\eta_{1,1,6},\ \eta_{1,2,1},\ \eta_{2,1,1},\ \eta_{2,2,6},\ \eta_{2,3,1},\ \eta_{3,2,1},\ \eta_{3,3,1},\ \eta_{3,3,6},$$

where $|\mathcal{G}| = 13$ and the largest relation found is of degree 7.

Characteristic 11: For the splitting field $\mathbb{F}_{11}$ with degree of computation $n = 100$:

$$H^*\left(M_{23}, \mathbb{F}_{11}\right) \cong \mathbb{F}_{11}\left[x_9, x_{10}\right] / \langle x_9^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 13 generators:

$$\eta_{1,1,5},\ \eta_{1,1,10},\ \eta_{1,2,1},\ \eta_{1,5,1},\ \eta_{2,3,1},\ \eta_{2,3,2},\ \eta_{3,1,1},\ \eta_{3,4,4},\ \eta_{4,2,4},\ \eta_{4,5,1},\ \eta_{5,1,1},\ \eta_{5,4,1},\ \eta_{5,5,10},$$

where $|\mathcal{G}| = 30$ and the largest relation found is of degree 15.

Characteristic 23: For the splitting field $\mathbb{F}_{23}$ with degree of computation $n = 100$:

$$H^*\left(M_{23}, \mathbb{F}_{23}\right) \cong \mathbb{F}_{23}\left[x_{21}, x_{22}\right] / \langle x_{21}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 35 generators:

$$\eta_{1,3,4},\ \eta_{1,8,1},\ \eta_{2,4,1},\ \eta_{2,7,6},\ \eta_{3,9,3},\ \eta_{3,11,1},\ \eta_{4,2,1},\ \eta_{4,5,1},\ \eta_{4,10,6},\eta_{5,4,1},\ \eta_{5,5,22},\ \eta_{5,8,1},$$

$$\eta_{5,8,8},\ \eta_{6,3,1},\eta_{6,5,1},\ \eta_{6,5,8},\ \eta_{6,6,2},\ \eta_{6,8,7},\ \eta_{6,8,14},\ \eta_{6,11,3},\ \eta_{7,1,3},\ \eta_{7,10,1},\ \eta_{8,6,1},\eta_{8,6,8},$$

$$\eta_{8,8,2},\ \eta_{8,10,1},\ \eta_{9,2,6},\eta_{9,11,1},\ \eta_{10,1,1},\ \eta_{10,7,1},\ \eta_{10,8,3},\ \eta_{10,11,10},\ \eta_{11,4,6},\ \eta_{11,6,1},\ \eta_{11,9,1},$$

where $|\mathcal{G}| = 105$ and the largest relation found is of degree 31.

5.5.7 *HS*

The order of the Higman-Sims group is $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 = 44352000$.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 50$:

$$H^*\left(HS, \mathbb{F}_3\right) \cong \mathbb{F}_3\left[x_7, x_8, x_{10}, x_{11}, x_{12}, x_{15}, x_{16}\right] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is the set:

$$x_7x_7, x_7x_{10}, x_7y_{11}, x_8x_{10} + 2 \cdot x_7x_{11}, x_8y_{11} + 2 \cdot x_7x_{12}, x_{10}x_{10},$$
$$x_{10}x_{11}, x_{10}y_{11}, x_{10}x_{12} + 2 \cdot x_7x_{15}, x_{10}x_{15}, x_{11}x_{11},$$
$$x_{11}y_{11} + x_7x_{15}, x_{11}x_{12} + 2 \cdot x_8x_{15}, x_{11}x_{15}, y_{11}y_{11}, y_{11}x_{12} + 2 \cdot x_7x_{16},$$
$$y_{11}x_{15} + 2 \cdot x_{10}x_{16}, x_{12}x_{12} + 2 \cdot x_8x_{16}, x_{12}x_{15} + 2 \cdot x_{11}x_{16}, x_{15}x_{15}.$$

The number of generators is 8 and $|\mathcal{G}| = 20$.

The Ext-algebra computation for $n = 30$ produces the following 50 generators:

$$\eta_{1,1,8},\ \eta_{1,2,3},\ \eta_{1,2,8},\ \eta_{1,3,1},\ \eta_{1,3,6},\ \eta_{1,5,1},\ \eta_{2,1,3},\ \eta_{2,1,8},\ \eta_{2,3,1},\ \eta_{2,3,6},\ \eta_{2,5,1},\ \eta_{2,5,6},\ \eta_{2,6,3},$$
$$\eta_{2,6,4},\ \eta_{2,6,8},\ \eta_{3,1,1},\ \eta_{3,1,6},\ \eta_{3,2,1},\ \eta_{3,2,6},\ \eta_{3,3,3},\ \xi_{3,3,3},\ \eta_{3,3,4},\ \eta_{3,3,8},\ \xi_{3,3,8},\ \eta_{3,4,1},$$
$$\eta_{3,5,3},\ \eta_{3,5,8},\ \eta_{3,6,1},\ \eta_{3,6,6},\ \eta_{3,7,1},\ \eta_{4,3,1},\ \eta_{4,7,3},\ \eta_{4,7,8},\ \eta_{5,1,1},\ \eta_{5,2,1},\ \eta_{5,2,6},\ \eta_{5,3,3},$$
$$\eta_{5,3,8},\ \eta_{5,5,8},\ \eta 5,5,11,\ \eta_{5,7,1},\ \eta_{6,2,3},\ \eta_{6,2,4},\ \eta_{6,2,8},\ \eta_{6,3,1},\ \eta_{6,3,6},\ \eta_{7,3,1},\ \eta_{7,4,3},\ \eta_{7,4,8},\ \eta_{7,5,1},$$

where $|\mathcal{G}| = 426$ and the largest relation found is of degree 30.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 30$:

$$H^*(HS, \mathbb{F}_5) \cong$$
$$\mathbb{F}_5[x_4, x_5, x_7, y_7, x_8, y_8, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{23}, x_{24}, x_{27}, x_{28}] / \langle \mathcal{G} \rangle.$$

The number of generators is 16 and $|\mathcal{G}| = 57$.

The Ext-algebra computation for $n = 8$ produces 189 generators where the largest generator found is of degree 8 and $|\mathcal{G}| = 1945$.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(HS, \mathbb{F}_7) \cong \mathbb{F}_7[x_{11}, x_{12},] / \langle x_{11}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 16 generators:

$$\eta_{1,2,1},\ \eta_{1,5,1},\ \eta_{1,6,6},\ \eta_{2,1,1},\ \eta_{2,4,6}, \eta_{3,5,1},\ \eta_{3,5,6},\ \eta_{3,6,1},$$

$$\eta_{4,2,6},\ \eta_{4,6,1}, \eta_{5,1,1},\ \eta_{5,3,1},\ \eta_{5,3,6},\ \eta_{6,1,6},\ \eta_{6,3,1},\ \eta_{6,4,1},$$

where $|\mathcal{G}| = 30$ and the largest relation found is of degree 7.

Characteristic 11: For the splitting field $\mathbb{F}_{11}$ with degree of computation $n = 100$:

$$H^*(HS, \mathbb{F}_{11}) \cong \mathbb{F}_{11}[x_9, x_{10}] / \langle x_9^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 14 generators:

$$\eta_{1,1,10},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,10},\ \eta_{3,1,1},$$

$$\eta_{3,3,10},\ \eta_{3,5,1},\ \eta_{4,4,10},\ \eta_{4,5,1},\ \eta_{5,3,1},\ \eta_{5,4,1},\ \eta_{5,5,10},$$

where $|\mathcal{G}| = 34$ and the largest relation found is of degree 11.

5.5.8 J_3

The order of the Janko group J_3 is $2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19 = 50232960$.

Characteristic 3: For the splitting field $\mathbb{F}_9$ with degree of computation $n = 14$:

$$H^*(J_3, \mathbb{F}_9) \cong$$
$$\mathbb{F}_9[x_2, x_3, y_3, x_4, x_7, y_7, z_7, w_7, x_8, y_8, z_8, x_{11}, y_{11}, x_{12}, y_{12}, z_{12}, w_{12}, x_{13}, y_{13}] / \langle \mathcal{G} \rangle.$$

The number of generators is 19 and $|\mathcal{G}| = 50$.

Characteristic 5: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 100$:

$$H^*(J_3, \mathbb{F}_5) \cong \mathbb{F}_5[x_3, x_4]/\langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 5 generators:

$$\eta_{1,1,1},\ \eta_{1,1,4},\ \eta_{1,2,1},\ \eta_{2,1,1},\ \eta_{2,2,4},$$

where $|\mathcal{G}| = 8$ and the largest relation found is of degree 5.

Characteristic 17: For the splitting field $\mathbb{F}_{17}$ with degree of computation $n = 100$:

$$H^*(J_3, \mathbb{F}_{17}) \cong \mathbb{F}_{17}[x_{15}, x_{16}]/\langle x_{15}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 23 generators:

$$\eta_{1,3,8},\ \eta_{1,4,1},\ \eta_{2,5,8},\ \eta_{2,7,1},\ \eta_{2,8,1},\ \eta_{3,1,8},\ \eta_{3,8,1},\ \eta_{4,1,1},\ \eta_{4,5,1},\ \eta_{4,8,8}, \eta_{5,2,8},\ \eta_{5,4,1},$$

$$\eta_{5,6,1},\ \eta_{6,5,1},\ \eta_{6,6,7}, \eta_{6,6,16},\ \eta_{6,7,1},\ \eta_{7,2,1},\ \eta_{7,6,1},\ \eta_{7,7,16},\ \eta_{8,2,1},\ \eta_{8,3,1},\ \eta_{8,4,8},$$

where $|\mathcal{G}| = 58$ and the largest relation found is of degree 23.

Characteristic 19: For the splitting field $\mathbb{F}_{19}$ with degree of computation $n = 100$:

$$H^*(J_3, \mathbb{F}_{19}) \cong \mathbb{F}_{19}[x_{17}, x_{18}]/\langle x_{17}^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 26 generators:

$$\eta_{1,1,18},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,18},\ \eta_{2,8,1},\ \eta_{3,1,1},\ \eta_{3,3,18},\ \eta_{3,9,1},\ \eta_{4,4,18},\ \eta_{4,7,1},\ \eta_{4,8,1},\ \eta_{5,5,18},$$

$$\eta_{5,9,1},\ \eta_{6,6,1},\ \eta_{6,6,18},\ \eta_{6,7,1},\ \eta_{7,4,1},\ \eta_{7,6,1},\ \eta_{7,7,18},\ \eta_{8,2,1},\ \eta_{8,4,1},\ \eta_{8,8,18},\ \eta_{9,3,1},\ \eta_{9,5,1},\ \eta_{9,9,18},$$

where $|\mathcal{G}| = 78$ and the largest relation found is of degree 19.

5.5.9 McL

The order of the McLaughlin group is $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 = 898128000$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 20$:

$$H^*(McL, \mathbb{F}_2) \cong \mathbb{F}_2[x_7, x_8, x_{11}, x_{12}, x_{14}, y_{14}, x_{15}, y_{15}, x_{17}, x_{18}, y_{18}]]/\langle x_7^2, x_7x_{11}\rangle.$$

The number of generators is 11 and $|\mathcal{G}| = 2$.

The Ext-algebra computation for $n = 8$ produces 148 generators where the largest generator is of degree 8 and $|\mathcal{G}| = 1454$.

Characteristic 5: For the splitting field $\mathbb{F}_{25}$ with degree of computation $n = 40$:

$$H^*(McL, \mathbb{F}_{25}) \cong \mathbb{F}_{25}[x_4, x_5, x_7, x_8, x_{13}, x_{14}, x_{15}, x_{16}, x_{23}, x_{24}, x_{39}]/\langle \mathcal{G}\rangle.$$

The number of generators is 11 and $|\mathcal{G}| = 42$.

The Ext-algebra computation for $n = 14$ produces 295 generators where the largest generator is of degree 14 and $|\mathcal{G}| = 4988$.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(McL, \mathbb{F}_7) \cong \mathbb{F}_7[x_5, x_6]/\langle x_5^2\rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 8 generators:

$$\eta_{1,1,3},\ \eta_{1,1,6},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,3,3},\ \eta_{3,1,1},\ \eta_{3,2,3},$$

where $|\mathcal{G}| = 17$ and the largest relation found is of degree 9.

Characteristic 11: For the splitting field $\mathbb{F}_{11}$ with degree of computation $n = 100$:

$$H^*(McL, \mathbb{F}_{11}) \cong \mathbb{F}_{11}[x_9, x_{10}] / \langle x_9^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 14 generators:

$$\eta_{1,1,10},\ \eta_{1,4,1},\ \eta_{1,5,1},\ \eta_{2,2,10},\ \eta_{2,4,1},\ \eta_{3,3,1},\ \eta_{3,3,10},$$

$$\eta_{3,5,1},\ \eta_{4,1,1},\ \eta_{4,2,1},\ \eta_{4,4,10},\ \eta_{5,1,1},\ \eta_{5,3,1},\ \eta_{5,5,10},$$

where $|\mathcal{G}| = 30$ and the largest relation found is of degree 11.

5.6 Classical Groups

5.6.1 $L_2(7)$

The order of $L_2(7)$ is $2^3 \cdot 3 \cdot 7 = 168$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 50$:

$$H^*(L_2(7), \mathbb{F}_2) \cong \mathbb{F}_2[x_2, x_3, y_3] / \langle x_3 y_3 \rangle.$$

The number of generators is 3 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 40$ produces the following 6 generators:

$$\eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,3,1},\ \eta_{3,1,1},\ \eta_{3,2,1},$$

where $|\mathcal{G}| = 44$ and the largest relation found is of degree 40.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 100$:

$$H^*(L_2(7), \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4] / \langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 4 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{2,1,1},\ \eta_{2,1,2},$$

and the set $\mathcal{G}$ is:

$$\eta_{2,1,1}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,1},\ \eta_{2,1,1}\eta_{1,2,2} + 2 \cdot \eta_{2,1,2}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,2} + 2 \cdot \eta_{1,2,2}\eta_{2,1,1}.$$

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*\left(L_2\left(7\right), \mathbb{F}_7\right) \cong \mathbb{F}_7\left[x_5, x_6\right]/\langle x_5^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 8 generators:

$$\eta_{1,1,1},\ \eta_{1,1,6},\ \eta_{1,2,1},\ \eta_{2,1,1},\ \eta_{2,2,6},\ \eta_{2,3,1},\ \eta_{3,2,1},\ \eta_{3,3,6},$$

where $|\mathcal{G}| = 16$ and the largest relation found is of degree 7.

5.6.2 $L_3\left(3\right)$

The order of $L_3\left(3\right)$ is $2^4 \cdot 3^3 \cdot 13 = 5616$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 40$:

$$H^*\left(L_3\left(3\right), \mathbb{F}_2\right) \cong \mathbb{F}_2\left[x_3, x_4, y_5\right]/\langle x_3^2 x_4 + x_5^2 \rangle.$$

The number of generators is 3 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for 40 produces the following 8 generators:

$$\eta_{1,1,4},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,4},\ \eta_{3,1,1},\ \eta_{3,3,1},$$

where $|\mathcal{G}| = 32$ and the largest relation found is of degree 40.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 30$:

$$H^*(L_3(3), \mathbb{F}_3) \cong \mathbb{F}_3[x_3, x_4, y_4, x_5, x_9, x_{10}, y_{10}, z_{10}, x_{11}, y_{11}, z_{11}, x_{12}, x_{15}, y_{15}, x_{16}, y_{16}] / \langle \mathcal{G} \rangle.$$

The number of generators is 16 and $|\mathcal{G}| = 101$.

The Ext-algebra computation for $n = 30$ produces 60 generators where the largest generator found is of degree 12. $|\mathcal{G}| = 787$ and the largest relation found is of degree 30.

Characteristic 13: For the splitting field $\mathbb{F}_{13}$ with degree of computation $n = 100$:

$$H^*(L_3(3), \mathbb{F}_{13}) \cong \mathbb{F}_{13}[x_5, x_6] / \langle x_5^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 8 generators:

$$\eta_{1,1,6},\ \eta_{1,2,1},\ \eta_{2,1,1},\ \eta_{2,2,6},\ \eta_{2,3,1},\ \eta_{3,2,1},\ \eta_{3,3,1},\ \eta_{3,3,6},$$

where $|\mathcal{G}| = 13$ and the largest relation found is of degree 7.

5.6.3 $L_2(8)$

The order of $L_2(8)$ is $2^3 \cdot 3^2 \cdot 7 = 504$.

Characteristic 2: For the splitting field $\mathbb{F}_8$ with degree of computation $n = 20$:

$$H^*(L_2(8), \mathbb{F}_8) \cong \mathbb{F}_8[x_3, x_4, y_4, z_4, x_5, y_5, z_5, x_6, y_6, z_6, x_7, y_7, z_7] / \langle \mathcal{G} \rangle.$$

The number of generators is 13 and $|\mathcal{G}| = 54$.

The Ext-algebra computation for $n = 20$ produces 55 generators where the largest generator is of degree 7. $|\mathcal{G}| = 498$ where the largest relation found is of degree 14.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 100$:

$$H^*\left(L_2\left(8\right), \mathbb{F}_3\right) \cong \mathbb{F}_3\left[x_3, x_4\right] / \langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 5 generators:

$$\eta_{1,1,4},\ \eta_{1,2,1},\ \eta_{2,1,1},\ \eta_{2,2,1},\ \eta_{2,2,4},$$

where $|\mathcal{G}| = 7$ and the largest relation found is of degree 5.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*\left(L_2\left(8\right), \mathbb{F}_7\right) \cong \mathbb{F}_7\left[x_3, x_4\right] / \langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 4 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{2,1,1},\ \eta_{2,1,2},$$

where $\mathcal{G}$ is the set:

$$\eta_{2,1,1}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,1},\ \eta_{2,1,1}\eta_{1,2,2} + Z(7)^3 \cdot \eta_{2,1,2}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,2} + Z(7)^3 \cdot \eta_{1,2,2}\eta_{2,1,1}.$$

5.6.4 $U_3\left(3\right)$

The order of $U_3\left(3\right)$ is $2^5 \cdot 3^3 \cdot 7 = 6048$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 30$:

$$H^*\left(U_3\left(3\right), \mathbb{F}_2\right) \cong \mathbb{F}_2\left[x_3, x_4, x_5, x_6\right] / \langle x_3^2, x_5^2 \rangle.$$

The number of generators is 4 and $|\mathcal{G}| = 2$.

The Ext-algebra computation for $n = 30$ produces the following 16 generators:

$$\eta_{1,1,3},\ \eta_{1,1,4},\ \eta_{1,1,5},\ \eta_{1,1,6},\ \eta_{1,2,1}, \eta_{1,2,2},\ \eta_{2,1,1},\ \eta_{2,1,2},$$

$$\eta_{2,3,1},\ \eta_{2,3,2},\ \xi_{2,3,2},\ \eta_{2,3,3},\ \eta_{3,2,1},\ \eta_{3,2,2},\ \xi_{3,2,2},\ \eta_{3,2,3},$$

where $|\mathcal{G}| = 91$ and the largest relation found is of degree 30.

Characteristic 3: For the splitting field $\mathbb{F}_9$ with degree of computation $n = 40$:

$$H^*(U_3(3), \mathbb{F}_9) \cong \mathbb{F}_9[x_3, x_4, y_4, x_5, x_9, x_{10}, x_{11}, x_{12}] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is given by the set:

$$x_3^2, x_3y_4, x_3x_9, x_4y_4 + Z(3) \cdot x_3x_5, x_4x_9 + Z(3^2)^2 \cdot x_3x_{10}, y_4^2,$$
$$y_4x_5, y_4x_9, y_4x_{10} + Z(3^2) \cdot x_3x_{11}, y_4x_{11}, x_5^2, x_5x_9 + Z(3^2)^3 \cdot x_3x_{11},$$
$$x_5x_{10} + Z(3^2) \cdot x_4x_{11}, x_5x_{11}, x_3x_4x_{12} + Z(3) \cdot x_9x_{10}, x_3x_5x_{12} + Z(3^2) \cdot x_9x_{11},$$
$$x_4^2x_{12} + Z(3^2)^5 \cdot x_9x_{11} + Z(3^2)^2 \cdot x_{10}^2, x_4x_5x_{12} + Z(3^2)^7 \cdot x_{10}x_{11}, x_9^2, x_{11}^2.$$

The number of generators is 8 and $|\mathcal{G}| = 20$.

The Ext-algebra computation for $n = 40$ produces 66 generators where the largest generator found is of degree 12. $|\mathcal{G}| = 560$ and the largest relation found is of degree 23.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^*(U_3(3), \mathbb{F}_7) \cong \mathbb{F}_7[x_5, x_6] / \langle x_5^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 8 generators:

$$\eta_{1,2,1},\ \eta_{1,3,3},\ \eta_{2,1,1},\ \eta_{2,2,3},\ \eta_{2,2,6},\ \eta_{2,3,1},\ \eta_{3,1,3},\ \eta_{3,2,1},$$

where $|\mathcal{G}| = 17$ and the largest relation found is of degree 9.

5.6.5 $U_3(4)$

The order of $U_3(4)$ is $2^6 \cdot 3 \cdot 5^2 \cdot 13 = 62400$.

Characteristic 2: For the splitting field $\mathbb{F}_{16}$ with degree of computation $n = 14$:

$$H^*\left(U_3(4), \mathbb{F}_{16}\right) \cong \mathbb{F}_{16}[x_5, y_5, z_5, x_6, y_6, z_6, w_6, x_7, y_7, x_8, y_8, z_8, w_8,$$
$$x_9, y_9, z_9, x_{11}, y_{11}, x_{12}, y_{12}, z_{12}, w_{12}, x_{13}, y_{13}]/\langle \mathcal{G} \rangle.$$

The number of generators is 24 and $|\mathcal{G}| = 81$.

The Ext-algebra computation for $n = 5$ produces 92 generators where the largest generator found is of degree 4 and $|\mathcal{G}| = 521$.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 100$:

$$H^*\left(U_3\left(4\right), \mathbb{F}_3\right) \cong \mathbb{F}_3\left[x_3, x_4\right]/\langle x_3^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 4 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{2,1,1},\ \eta_{2,1,2},$$

where $\mathcal{G}$ is the set:

$$\eta_{2,1,1}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,1},\ \eta_{2,1,1}\eta_{1,2,2} + 2 \cdot \eta_{2,1,2}\eta_{1,2,1},\ \eta_{1,2,1}\eta_{2,1,2} + 2 \cdot \eta_{1,2,2}\eta_{2,1,1}.$$

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 30$:

$$H^*\left(U_3\left(4\right), \mathbb{F}_5\right) \cong \mathbb{F}_5\left[x_3, x_4, x_5, x_6\right]/\langle x_3^2, x_5^2 \rangle.$$

The number of generators is 4 and $|\mathcal{G}| = 2$.

The Ext-algebra computation for $n = 20$ produces the following 19 generators:

$$\eta_{1,1,1},\ \eta_{1,1,3},\ \eta_{1,1,4},\ \eta_{1,1,6},\ \eta_{1,3,1},\ \eta_{1,3,2},\ \eta_{2,2,3},\ \eta_{2,2,4},\ \eta_{2,2,5},\ \eta_{2,2,6},$$
$$\eta_{2,3,1},\ \eta_{2,3,2},\ \eta_{3,1,1},\ \eta_{3,1,2},\ \eta_{3,2,1},\ \eta_{3,2,2},\ \eta_{3,3,4},\ \eta_{3,3,5},\ \eta_{3,3,6},$$

where $|\mathcal{G}| = 102$ and the largest relation found is of degree 20.

Characteristic 13: For the splitting field $\mathbb{F}_{13}$ with degree of computation $n = 100$:

$$H^*(U_3(4), \mathbb{F}_{13}) \cong \mathbb{F}_{13}[x_5, x_6] / \langle x_5^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 8 generators:

$$\eta_{1,1,3},\ \eta_{1,1,6},\ \eta_{1,2,1},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,3,3},\ \eta_{3,1,1},\ \eta_{3,2,3},$$

where $|\mathcal{G}| = 17$ and the largest relation found is of degree 9.

5.6.6 $U_3(5)$

The order of $U_3(5)$ is $2^4 \cdot 3^2 \cdot 5^3 \cdot 7 = 126000$.

Characteristic 2: For the splitting field $\mathbb{F}_2$ with degree of computation $n = 30$:

$$H^*(U_3(5), \mathbb{F}_2) \cong \mathbb{F}_2[x_3, x_4, x_5] / \langle x_3^2 x_4 + x_5^2 \rangle.$$

The number of generators is 3 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 30$ produces the following 11 generators:

$$\eta_{1,1,3},\ \eta_{1,1,4},\ \eta_{1,1,5},\ \eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{1,3,1},\ \eta_{2,1,1},\ \eta_{2,1,2},\ \eta_{2,2,4},\ \eta_{3,1,1},\ \eta_{3,3,3},$$

where $|\mathcal{G}| = 61$ and the largest relation found is of degree 30.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 30$:

$$H^*(U_3(5), \mathbb{F}_3) \cong \mathbb{F}_3[x_2, x_3, x_7, y_7, x_8, y_8, x_{11}, x_{12}] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is the set:

$$x_2^2, x_2x_3, x_2x_7, x_2y_7, x_2x_{11}, x_3^2, x_3x_7 + x_2x_8 + x_2y_8, x_3y_7 + 2 \cdot x_2x_8,$$

$$x_3x_{11} + 2 \cdot x_2x_{12}, x_7^2, x_7y_7 + x_2x_{12}, y_7^2, y_7y_8 + x_3x_{12} + x_7x_8 + y_7x_8, y_7x_{11} + x_7x_{11},$$

$$y_8x_{11} + y_7x_{12} + x_7x_{12}, x_8y_8^2 + x_{12}^2 + x_8^2y_8, x_2x_8y_8 + x_7x_{11} + x_2x_8^2, x_{11}^2,$$

$$x_3x_8y_8 + y_7x_{12} + x_8x_{11} + x_3x_8x_8, x_7x_8y_8 + 2 \cdot x_{11}x_{12} + 2 \cdot x_3x_8x_{12}$$

The number of generators is 8 and $|\mathcal{G}| = 20$.

The Ext-algebra computation for $n = 30$ produces the following 16 generators:

$$\eta_{1,2,1},\ \eta_{1,2,2},\ \eta_{2,1,1},\ \eta_{2,1,2},\ \eta_{2,3,1}, \eta_{2,3,2},\ \eta_{2,4,1},\ \eta_{2,4,2},$$

$$\eta_{2,5,1},\ \eta_{2,5,2}, \eta_{3,2,1},\ \eta_{3,2,2},\ \eta_{4,2,1},\ \eta_{4,2,2},\ \eta_{5,2,1},\ \eta_{5,2,2},$$

where $|\mathcal{G}| = 64$ and the largest relation found is of degree 30.

Characteristic 5: For the splitting field $\mathbb{F}_{25}$ with degree of computation $n = 20$:

$$H^* (U_3 (5), \mathbb{F}_{25}) \cong \mathbb{F}_{25} [x_4, y_4, x_5, y_5, x_7, y_7, z_7, x_8, y_8, z_8, x_{13}, y_{13}, x_{14}, y_{14}, x_{15}, x_{16}] / \langle \mathcal{G} \rangle.$$

The number of generators is 16 and $|\mathcal{G}| = 65$.

The Ext-algebra computation for $n = 6$ produces 51 generators where the largest generator found is of degree 5 and $|\mathcal{G}| = 409$.

Characteristic 7: For the splitting field $\mathbb{F}_7$ with degree of computation $n = 100$:

$$H^* (U_3 (5), \mathbb{F}_7) \cong \mathbb{F}_7 [x_5, x_6] / \langle x_5^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 8 generators:

$$\eta_{1,2,1},\ \eta_{1,3,3},\ \eta_{2,1,1},\ \eta_{2,2,3},\ \eta_{2,2,6},\ \eta_{2,3,1},\ \eta_{3,1,3},\ \eta_{3,2,1},$$

where $|\mathcal{G}| = 17$ and the largest relation found is of degree 9.

5.6.7 $U_4 (2)$

The order of $U_4 (2)$ is $2^6 \cdot 3^4 \cdot 5 = 25920$.

Characteristic 2: For the splitting field $\mathbb{F}_4$ with degree of computation $n = 14$:

$$H^*\left(U_4\left(2\right), \mathbb{F}_4\right) \cong \mathbb{F}_4\left[x_2, x_3, y_3, x_4, x_5, x_{10}\right] / \langle \mathcal{G} \rangle.$$

where $\mathcal{G}$ is the set:

$$y_3^2 + x_3 y_3,\ y_3 x_5,\ y_3 x_{10} + Z(2^2) \cdot x_2^5 y_3 + Z(2^2) \cdot x_2 y_3 x_4^2$$

The number of generators is 6 and $|\mathcal{G}| = 3$.

The Ext-algebra computation for $n = 10$ produces 80 generators where the largest generator found is of degree 7 and $|\mathcal{G}| = 804$.

Characteristic 3: For the splitting field $\mathbb{F}_3$ with degree of computation $n = 20$:

$$H^*\left(U_4\left(2\right), \mathbb{F}_4\right) \cong \mathbb{F}_4\left[x_3, x_4, x_5, x_6, x_7, x_8\right] / \langle x_3^2, x_5^2, x_7^2 \rangle.$$

The number of generators is 6 and $|\mathcal{G}| = 3$.

The Ext-algebra computation for $n = 10$ produces 81 generators where the largest generator found is of degree 9 and $|\mathcal{G}| = 984$.

Characteristic 5: For the splitting field $\mathbb{F}_5$ with degree of computation $n = 100$:

$$H^*\left(U_4\left(2\right), \mathbb{F}_5\right) \cong \mathbb{F}_5\left[x_7, x_8\right] / \langle x_7^2 \rangle.$$

The number of generators is 2 and $|\mathcal{G}| = 1$.

The Ext-algebra computation for $n = 100$ produces the following 10 generators:

$$\eta_{1,2,1},\ \eta_{1,2,4},\ \eta_{1,4,1},\ \eta_{2,1,1},\ \eta_{2,1,4},\ \eta_{2,3,1},\ \eta_{3,2,1},\ \eta_{3,4,4},\ \eta_{4,1,1},\ \eta_{4,3,4},$$

where $|\mathcal{G}| = 16$ and the largest relation found is of degree 5.

5.7 Timing Comparisons of Projective Resolutions

In this section we provide a comparison of timings of minimal projective resolutions for basic algebras arising from a variety of groups. We compare the method using the author's implementation using linear algebra (Lin), the author's Anick-Green resolution in GAP (Ani), and Green and Feustel's program GRB [FG91]. Note that the program GRB only works over fields of size p and no extensions which are needed for groups such as A_4 in characteristic 2. We use the notation `A` in our table of timings to denote when the program GRB aborted due to memory issues. It is clear from the results that we obtained that due to the poor performance in terms of timings from the Anick-Green methods, perhaps due to the need to compute Gröbner bases and then reduce rather large matrices repeatedly without linear algebra, that the linear algebra method is vastly superior in terms of speed. We also found that for the small examples below that we computed here that there were no memory savings. After these initial findings, we did not make any significant attempts to speed up our Anick-Green implementation in GAP, as the stand alone C-program GRB could not even compete with the author's linear algebra implementation in GAP.

We must note, however, that there are advantages to the method of Anick and Green. The first is that their method only relies on having an Artinian ring. Therefore if we wish to compute the Ext-algebra for an Artinian ring which is infinite dimensional with an admissible order such that there is a finite Gröbner basis $\mathcal{G}$ the method of linear algebra clearly does not work. However if we have the conditions above, we can still use the Anick-Green method.

In the comparisons, we include a variety of alternating groups, symmetric groups, sporadic groups, linear groups, and also p-groups of order 16 and 27. We label the nonabelian p-groups by their position in the list of small groups in the library of small groups in `GAP`. For all of the computations, we give the group name G, the splitting field $\mathbb{F}_q$ for the group algebra $\Bbbk G$, the program used, and timings for projective resolutions for all of the simple $\Bbbk G$-modules S_i.

G	$\mathbb{F}_q$	Prg	1	2	3	4	5	6	7
A_4	$\mathbb{F}_4$	Lin	30	20	20				
		Ani	100	100	90				
		GRB	-	-	-				
	$\mathbb{F}_3$	Lin	0						
		Ani	10						
		GRB	0						
A_5	$\mathbb{F}_4$	Lin	10	10	10				
		Ani	80	20	30				
		GRB	-	-	-				
	$\mathbb{F}_3$	Lin	0	0					
		Ani	10	10					
		GRB	0	0					
A_6	$\mathbb{F}_2$	Lin	10	0	10				
		Ani	90	20	40				
		GRB	10	10	10				
	$\mathbb{F}_9$	Lin	50	30	20	50			
		Ani	1080	5310	1410	14160			
		GRB	-	-	-	-			
A_7	$\mathbb{F}_2$	Lin	10	20	0				
		Ani	130	60	20				
		GRB	10	10	10				
	$\mathbb{F}_9$	Lin	70	50	40	50			
		Ani	15980	9320	10970	11370			
		GRB	-	-	-	-			
A_8	$\mathbb{F}_2$	Lin	880	2160	960	840	1340	920	1010
		GRB	369040	A	158000	88540	A	212150	185910
	$\mathbb{F}_3$	Lin	30	20	80	30	30		
		Ani	2240	2040	10460	2040	2280		
		GRB	10	10	10	10	10		

TABLE 5.3. Minimal Resolution Comparisons: Alternating Groups

G	$\mathbb{F}_q$	Prg	1	2	3	4	5
S_4	$\mathbb{F}_2$	Lin	10	20			
		Ani	90	190			
		GRB	10	10			
	$\mathbb{F}_3$	Lin	10	0			
		Ani	10	10			
		GRB	10	10			
S_5	$\mathbb{F}_2$	Lin	10	10			
		Ani	320	20			
		GRB	10	10			
	$\mathbb{F}_3$	Lin	0	0			
		Ani	10	0			
		GRB	0	0			
S_6	$\mathbb{F}_2$	Lin	260	70	70		
		Ani	3883020	62150	54500		
		GRB	104760	1000	330		
	$\mathbb{F}_3$	Lin	20	20	40	30	30
		Ani	1510	1650	4630	2500	3120
		GRB	10	10	10	10	10
S_7	$\mathbb{F}_2$	Lin	270	260	30		
		Ani	70780	177500	3620		
		GRB	430	420	60		
	$\mathbb{F}_3$	Lin	40	30	30	40	80
		Ani	3480	6020	4410	3930	16270
		GRB	10	10	10	10	10
S_8	$\mathbb{F}_2$	Lin	1160	3170	3880	4000	1470
		GRB	A	A	A	A	A
	$\mathbb{F}_3$	Lin	20	70	30	30	20
		Ani	3040	11360	2110	2430	2410
		GRB	10	10	10	10	10

TABLE 5.4. Minimal Resolution Comparisons: Symmetric Groups

G	$\mathbb{F}_q$	Prg	1	2	3	4	5	6	7	8
M_{11}	$\mathbb{F}_2$	Lin	10	0	20					
		Ani	190	130	380					
		GRB	10	10	10					
J_1	$\mathbb{F}_4$	Lin	180	30	70	80	120			
		Ani	15650	2790	15780	180100	35680			
		GRB	-	-	-	-	-			
	$\mathbb{F}_3$	Lin	0	10						
		Ani	10	10						
		GRB	0	10						
$U_3(3)$	$\mathbb{F}_2$	Lin	30	70	60					
		Ani	18870	97340	106000					
		GRB	570	2810	3440					
	$\mathbb{F}_9$	Lin	170	480	140	140	140	140	130	130
		Ani	422010							
		GRB	-	-	-	-	-	-	-	-
$L_3(3)$	$\mathbb{F}_2$	Lin	10	10	10					
		Ani	180	120	390					
		GRB	10	10	10					
	$\mathbb{F}_3$	Lin	210	110	90	90	80	80	90	80
		Ani	935080	224990	83490	104400	31570	29950	35090	37140
		GRB	810	10	10	10	10	10	10	140
$L_2(7)$	$\mathbb{F}_2$	Lin	10	10	10					
		Ani	310	230	230					
		GRB	10	10	10					
	$\mathbb{F}_3$	Lin	0	0						
		Ani	10	0						
		GRB	10	0						

TABLE 5.5. Minimal Resolution Comparisons: Other Groups

G	$\mathbb{F}_q$	Prg	1
16_1	$\mathbb{F}_2$	Lin	80
		Ani	6410
		GRB	5400
16_2	$\mathbb{F}_2$	Lin	20
		Ani	580
		GRB	20
16_3	$\mathbb{F}_2$	Lin	20
		Ani	8620
		GRB	1400
16_4	$\mathbb{F}_2$	Lin	40
		Ani	810
		GRB	60
16_5	$\mathbb{F}_2$	Lin	10
		Ani	7090
		GRB	10
16_6	$\mathbb{F}_2$	Lin	10
		Ani	3890
		GRB	230
16_7	$\mathbb{F}_2$	Lin	270
		Ani	29284670
		GRB	1073260
16_8	$\mathbb{F}_2$	Lin	80
		Ani	331390
		GRB	143010
16_9	$\mathbb{F}_2$	Lin	80
		Ani	2068240
		GRB	50890
27_1	$\mathbb{F}_3$	Lin	170
		Ani	69024150
		GRB	1254790
27_2	$\mathbb{F}_3$	Lin	30
		Ani	542560
		GRB	16140

TABLE 5.6. Minimal Resolution Comparisons: p-Groups

Appendix A
Timings

In this Appendix, we include the timings for the computations that we completed using our the programs for the implementations of the algorithms given in this dissertation in GAP. All of the computations were done using `GAP4r4` on an `AMD Opteron X86_64` 2 gigahertz processor with 8 gigabytes of RAM. The operating system is `Linux 2.4.24`. All timings are given in terms of milliseconds (ms) where the operating system only records to the nearest 10 ms. All groups are referred to by their name in the Atlas of Finite Groups [CCN$^+$85].

A.1 Gröbner Basis Computations

In Chapter 3 we gave an algorithm to compute a Gröbner basis $\mathcal{G}$ for a basic algebra B. Below we give timings and results of the computations done in GAP. The information that we provide is the name of the group G that corresponds to the basic algebra B, the characteristic of the splitting field for B (Prime), the time in milliseconds (ms) as timed in GAP, the size of the Gröbner basis $\mathcal{G}$, and the dimension of the basic algebra B.

Group	Prime	Time (ms)	Size $\mathcal{G}$	$\mathrm{Dim}_{\Bbbk} B$
A_4	2	10	11	12
	3	0	1	3
A_5	2	10	7	18
	3	10	2	6
	5	0	5	7
A_6	2	10	7	34
	3	10	31	36
	5	0	5	7
A_7	2	0	9	19
	3	20	39	36
	5	0	8	14
	7	0	9	11
A_8	2	4190	235	226
	3	10	30	46
	5	0	8	14
	7	10	8	11
A_9	2	9760	308	296
	3	5510	295	166
	5	0	8	14
	7	10	14	22
A_{10}	2	80370	549	646
	3	4860	281	166
	5	650	131	121
	7	10	16	22
A_{11}	2	173320	828	562
	3	24880	417	372
	5	420	108	121
	7	0	16	22
	11	10	15	19
A_{12}	3	12913600	3155	1781
	5	1950	189	178
	7	0	14	22
	11	0	15	19

TABLE A.1. Gröbner Basis Timings: Alternating Groups

Group	Prime	Time (ms)	Size $\mathcal{G}$	$\mathrm{Dim}_{\Bbbk} B$
S_3	2	0	1	2
	3	0	2	6
S_4	2	0	8	11
	3	0	2	6
S_5	2	0	5	19
	3	10	2	6
	5	0	8	14
S_6	2	70	58	68
	3	20	37	51
	5	10	8	14
S_7	2	10	28	38
	3	30	41	51
	5	10	8	14
	7	0	14	22
S_8	2	25670	463	289
	3	0	30	46
	5	0	10	14
	7	0	16	22
S_9	2	31010	487	370
	3	22370	426	332
	5	0	10	14
	7	0	14	22
S_{10}	2	2848870	2037	1292
	3	24980	440	332
	5	1970	186	183
	7	0	16	22
S_{11}	2	4797870	2581	1124
	3	27530	445	372
	5	1640	172	178
	7	0	16	22
	11	10	30	38

TABLE A.2. Gröbner Basis Timings: Symmetric Groups

Group	Prime	Time (ms)	Size $\mathcal{G}$	$\mathrm{Dim}_{\Bbbk} B$
M_{11}	2	0	16	22
	3	20	32	83
	5	0	4	20
	11	0	9	25
M_{12}	2	3290	249	134
	3	480	113	163
	5	0	8	14
	11	0	14	19
J_1	2	80	57	82
	3	0	2	6
	5	10	2	10
	7	10	14	22
	11	10	30	38
M_{22}	2	166230	750	799
	3	20	37	51
	5	10	8	14
	7	0	8	11
	11	0	9	29
J_2	2	829810	1305	1592
	3	1570	175	204
	5	250	102	72
	7	10	16	22
M_{23}	2	2373270	1879	1513
	3	20	33	81
	5	0	4	20
	7	0	8	11
	11	10	9	29
HS	2	11844960	2676	2462
	3	70	57	75
	5	91440	669	444
	7	0	14	22
	11	10	14	19
J_3	2	1087280	1455	1169
	3	1308710	1428	1754
	5	0	4	7
M_{24}	3	1670	175	213
	5	0	8	14
	7	0	9	11
McL	2	308650	923	1004
	5	330930	1056	788
	7	0	5	14

TABLE A.3. Gröbner Basis Timings: Sporadic Groups

Group	Prime	Time (ms)	Size $\mathcal{G}$	$\mathrm{Dim}_{\Bbbk} B$
$L_2(7)$	2	0	11	16
	3	0	2	6
	7	0	7	11
$L_3(3)$	2	10	16	22
	3	330	98	133
	13	0	8	13
$L_2(8)$	2	50	43	92
	3	0	5	9
	7	0	2	14
$U_3(3)$	2	10	21	108
	3	1190	166	145
	7	10	5	14
$U_3(4)$	2	1638310	1700	1306
	3	10	2	6
	5	10	19	72
	13	0	5	22
$U_3(5)$	2	10	10	67
	3	100	27	41
	5	33920	480	279
	7	0	5	14
$U_4(2)$	2	12370	329	318
	3	3900	241	163
	5	0	8	14

TABLE A.4. Gröbner Basis Timings: Classical Groups

A.2 Projective Resolutions

In the next pages we give timing comparisons for the projective resolutions up to degree $n = 20$ for many alternating groups, symmetric groups, sporadic groups, and classical groups. The method used is the linear algebra approach which we discovered was far superior to the Anick-Green Gröbner basis method in terms of timings. We make these computations for each PIM (for only the first 10 PIMs if there are more than 10) and record the timing in milliseconds for each of the PIMs given by its number according to its position in the basic algebra B.

G	p	1	2	3	4	5	6	7	8	9	10
A_4	2	630	620	630							
	3	0									
A_5	2	190	20	40							
	3	20	10								
	5	20	20								
A_6	2	160	10	20							
	3	120	1750	110	2240						
	5	10	10								
A_7	2	200	350	10							
	3	1340	1260	1300	1300						
	5	10	10	20	10						
	7	10	20	20							
A_8	2	397720	1070800	168270	168850	264150	169770	170400			
	3	580	560	1970	580	550					
	5	10	20	10	10						
	7	10	20	10							
A_9	2	308140	217590	90790	497650	4370	4260	275830			
	3	80540	56670	41840	256500	147790					
	5	20	10	10	0						
	7	30	20	30	20	20	10				
A_{10}	2	11348970	2509790	305550	2079880	382260	13680	501300			
	3	63060	331389	491620	201470	91990					
	5	1120	2340	4330	2710	290	280	2000	2120	2080	10130
	7	10	20	30	20	30	20				
A_{11}	3	79080	596760	97230	613110	79870	227080	70270	212750	628230	339620
	5	1340	11120	2260	2510	4540	2270	2710	1190	1190	1940
	7	20	30	10	20	10	20				
	11	0	40	30	30	40					

TABLE A.5. Minimal Resolution Timings: Alternating Groups

G	p	1	2	3	4	5	6	7	8	9	10
S_4	2	320	260								
	3	0	10								
S_5	2	320	10								
	3	0	10								
	5	10	10	10	20						
S_6	2	28260	1890	1840							
	3	370	380	530	90	440					
	5	10	10	10	20						
S_7	2	38610	74660	930							
	3	710	670	660	730	2410					
	5	10	10	10	10						
	7	20	40	10	10	30	20				
S_8	2	299600	3250250	1744590	3958290	289750					
	3	500	1810	490	530	500					
	5	10	10	10	20						
	7	20	30	10	20	30	30				
S_9	2	8123680	11819320	5270	147590	10805520					
	3	79930	80590	61030	231130	43890	45530	61200	230830	138940	139560
	5	20	20	10	10						
	7	20	30	30	10	0	30				
S_{10}	3	66030	192420	313420	66260	318720	192460	463090	459820	94770	94610
	5	1190	2550	4600	4670	1180	2520	2860	820	2880	11560
	7	30	10	40	20	30	30				

TABLE A.6. Minimal Resolution Timings: Symmetric Groups

G	p	1	2	3	4	5	6	7	8	9	10
M_{11}	2	160	20	460							
	3	270	280	340	210	270	320	60			
	5	20	20	10	10						
	11	30	20	10	20	40					
M_{12}	2	101560	291560	57490							
	3	4530	4220	4410	160	1600	160	1620	7210		
	5	10	10	20	10						
	11	30	20	40	0	40					
J_1	2	3510	90	1640	1600	3310					
	3	10	10								
	5	10	10								
	7	20	20	20	20	20	20				
	11	50	70	60	20	20	50	60	30	50	40
M_{22}	2	3217160	2478970	3222610	2703110	8347930	7365690	7343930			
	3	80	430	360	440	370					
	5	20	20	10	10						
	7	20	20	10							
	11	30	20	20	30	20					
J_2	2	937200	6762970	823920	836750	2396800	2403510	940180			
	3	240	230	300	6460	280	6390	5410	7320		
	5	650	6150	400	5360	3760	1390				
	7	10	30	20	20	20	30				
M_{23}	2	3406940	1538750	377670	1820280	517260	919830	1110790	2253030	564970	
	3	1820	1820	480	570	460	490	490			
	5	10	10	10	10						
	7	10	10	30							
	11	30	10	10	10	30					

TABLE A.7. Minimal Resolution Timings: Sporadic Groups

G	p	1	2	3	4	5	6	7	8	9	10
HS	2	32934630	42673960	2363590	20791250	22958620	6242190				
	3	320	320	1360	80	400	290	80			
	5	326110	120600	19080	70380	69810	111960	111890	315770	118160	707500
	7	30	10	30	20	20	30				
	11	40	50	30	20	30					
J_3	2	181090	283520	283110	409450	288970	41010	294740	1853840	892770	897580
	3	14460180	11507400	11386880	1136510	269320	266860	467770	472420		
	5	10	10								
	17	20	50	10	30	50	50	60	30		
	19	60	80	50	110	20	120	100	100	30	
M_{24}	3	4510	3670	540	2830	2280	2170	2560			
	5	10	20	10	10						
	7	0	20	20							
	11	30	10	20	20	30	20	70	30	30	40
McL	2	842190	92460	2572940	394840	2586440	2203740	2306960	2314440		
	5	57190	56990	215850	81510	22260	106640	252510	83610	173630	64660
	7	10	10	10							

TABLE A.8. Minimal Resolution Timings: Sporadic Groups Cont.

G	p	1	2	3	4	5	6	7	8	9	10
$L_2(7)$	2	500	360	380							
	3	0	10								
	7	30	10	10							
$L_2(8)$	2	16930	1150	1110	1100	70	80	80			
	3	10	10								
	7	10	10								
$L_3(3)$	2	170	30	490							
	3	5340	2400	2070	1930	2080	1950	1960	2100		
	13	10	20	40							
$U_3(3)$	2	670	1670	1560							
	3	1990	2960	540	560	670	540	520	500		
	7	10	30	10							
$U_3(4)$	2	169280	213920	211440	214500	212830	191230	187860	188560	187620	25190
		25140	21250	21190	21140	21080					
	3	10	20								
	5	2920	690	1820							
	13	20	10	20							
$U_3(5)$	2	170	30	50							
	3	520	1850	520	530	530					
	5	25680	112490	218970	13750	13850	31870	32210	152530		
	7	10	10	10							
$U_4(2)$	2	359750	1230410	96100	93800	218230	93190	93430			
	3	120400	62380	26800	33940	118040					
	5	20	20	10	10						

TABLE A.9. Minimal Resolution Timings: Classical Groups

A.3 Cohomology Ring

In this section, we include the results of timings for the computation $\dot{+}_{k=0}^{n} H^k(G, \Bbbk)$. Timings are all recorded in milliseconds. For each group G we list the prime p for the characteristic of the splitting field, degree n to which the calculation was completed, the time spent in finding the generators, the time spent rewriting the basis of $H^*(G, \Bbbk)$ as a basis in terms of the generators found, time spent computing a Gröbner basis, and the total time for all three steps in the calculation. It is assumed that a projective resolution for the trivial module has already been computed.

Group	Prime	n	Gen time	Spin Time	GB Time	Total
A_4	2	40	20250	660	90	21000
	3	100	110	640	10	760
A_5	2	100	75630	18110	3480	97220
	3	100	90	200	0	290
	5	100	90	200	0	290
A_6	2	40	6790	750	110	7650
	3	30	11770	280	200	12250
	5	100	90	200	10	300
A_7	2	30	3960	320	30	4310
	3	30	30290	220	200	30710
	5	100	70	60	0	130
	7	100	80	90	10	180
A_8	2	14	3355170	170	50	3355390
	3	30	7770	60	10	7840
	5	100	70	50	10	130
	7	100	90	90	0	180
A_9	2	14	5846890	180	70	5847140
	3	20	2187400	180	410	2187990
	5	100	70	60	0	130
	7	100	70	30	0	100
A_{10}	2	12	13480200	110	10	13480320
	3	20	3687140	170	380	3687690
	5	40	24380	100	60	24540
	7	100	70	30	0	100

TABLE A.10. Cohomology Ring Timings: Alternating Groups

Group	Prime	n	Gen time	Spin Time	GB Time	Total
S_4	2	100	610230	54510	13190	677930
	3	100	90	200	10	300
S_5	2	40	28920	3350	740	33010
	3	100	80	200	0	280
	5	100	70	60	0	130
S_6	2	20	6970350	4160	410	6974920
	3	50	20240	540	40	20820
	5	100	70	60	0	130
S_7	2	14	130560	330	20	130910
	3	40	23530	210	10	23750
	5	100	90	60	0	150
	7	100	70	30	0	100
S_8	2	12	79921250	800	120	79922170
	3	40	20930	180	20	21130
	5	100	80	60	0	140
	7	100	70	30	0	100
S_9	3	20	383910	50	10	383970
	5	100	70	60	10	140
	7	100	70	30	0	100
S_{10}	3	20	673030	60	10	673100
	5	30	3480	20	0	3500
	7	100	70	30	0	100

Table A.11. Cohomology Ring Timings: Symmetric Groups

Group	Prime	n	Gen time	Spin Time	GB Time	Total
M_{11}	2	100	105010	6530	560	112100
	3	100	236090	2310	1070	239470
	5	100	80	60	0	140
	11	100	80	30	10	120
M_{12}	2	12	1046820	70	30	1046920
	3	30	128000	300	1700	130000
	5	100	70	70	0	140
	11	100	80	40	0	120
J_1	2	30	261230	860	70	262160
	3	100	90	190	10	290
	5	100	100	190	10	300
	7	100	70	30	10	110
	11	100	60	20	0	80
	19	100	70	40	0	110
M_{22}	2	15	36335560	80	90	36335730
	3	50	40360	470	340	41170
	5	100	70	40	10	120
	7	100	70	80	0	150
	11	100	120	40	0	160
J_2	2	10	4813910	20	10	4813940
	3	30	136220	130	130	136480
	5	40	15930	120	0	16050
	7	100	60	30	0	90
M_{23}	2	14	358760	30	0	358790
	3	40	20950	90	50	21090
	5	100	80	60	0	140
	7	100	80	90	0	170
	11	100	80	50	0	130
	23	100	60	10	0	70
HS	3	50	21410	190	110	21710
	5	30	678010	120	270	678400
	7	100	80	30	0	110
	11	100	80	40	0	120
J_3	3	14	18812530	80	150	18812760
	5	100	80	160	10	250
	17	100	60	20	0	80
	19	100	60	10	0	70
McL	2	30	88676840	160	30	88677030
	5	40	401610	110	80	401900
	7	100	90	90	0	180
	11	100	70	50	0	120

Table A.12. Cohomology Ring Timings: Sporadic Groups

Group	Prime	n	Gen time	Spin Time	GB Time	Total
$L_2(7)$	2	50	39020	1430	180	40630
	3	100	130	160	0	290
	7	100	120	70	10	200
$L_3(3)$	2	40	4730	210	20	4960
	3	30	88140	240	1670	90050
	13	100	110	70	10	190
$L_2(8)$	2	20	134560	610	1300	136470
	3	100	90	200	0	290
	7	100	100	200	0	300
$U_3(3)$	2	30	39130	100	10	39240
	3	40	48090	340	230	48660
	7	100	70	90	10	170
$U_3(4)$	2	14	770580	50	440	771070
	3	100	80	210	0	290
	5	30	18060	110	0	18170
	13	100	110	90	0	200
$U_3(5)$	2	30	1640	80	10	1730
	3	30	12710	70	70	12850
	5	20	193390	50	370	193810
	7	100	80	120	10	210
$U_4(2)$	2	14	1542480	150	10	1542640
	3	20	544960	60	10	545030
	5	100	110	50	0	160

TABLE A.13. Cohomology Ring Timings: Classical Groups

A.4 Ext-Algebra

In this section we include timings for the Ext-algebra $E(\Bbbk G)$ up to a given degree n. Timings are all recorded in milliseconds. For each group G we list the prime p for the characteristic of the splitting field, degree n to which the calculation was completed, the time spent in finding the generators, the time spent rewriting the basis of $\dot{+}_{k=0}^{n} \dot{+}_{i,j} \operatorname{Ext}^k(S_i, S_j)$ as a basis in terms of the generators found, the time spent computing a Gröbner basis $\mathcal{G}$ and the total time for all three steps in the calculation. It is assumed that a projective resolution for all of the simple modules has already been computed.

Group	Prime	n	Gen time	Spin Time	GRB Time	Total
A_4	2	40	43970	556240	5360	605570
	3	100	110	4200	280	4590
A_5	2	100	86160	168530	24970	279660
	3	100	230	8850	620	9700
	5	100	390	8140	790	9320
A_6	2	40	8550	5460	1840	15850
	3	30	35190	6770	193020	234980
	5	100	460	8100	820	9380
A_7	2	40	28320	255510	10220	294050
	3	30	130040	26450	558770	715260
	5	100	900	18010	2330	21240
	7	100	960	13200	2000	16160
A_8	2	8	505680	7780	270590	784050
	3	30	111810	25910	103190	240910
	5	100	800	18110	2340	21250
	7	100	890	13330	1940	16160
A_9	3	15	23698550	23360	23693940	47415850
	5	100	790	18190	2550	21530
	7	100	1910	31170	9250	42330
A_{10}	3	15	51683270	32060	37055780	88771110
	5	20	204600	21090	20743630	20969320
	7	40	740	4100	1880	6720

TABLE A.14. Ext Algebra Timings: Alternating Groups

Group	Prime	n	Gen time	Spin Time	GRB Time	Total
S_4	2	40	38910	288880	8970	336760
	3	100	220	8930	530	9680
S_5	2	40	29620	15530	3930	49080
	3	100	220	9130	590	9940
	5	100	820	18860	2300	21980
S_6	2	20	7684750	19460	79620	7783830
	3	30	47570	11000	1010810	1069380
	5	30	250	1210	260	1720
S_7	2	10	52570	15510	6830	74910
	3	20	45840	7900	370940	424680
	5	30	250	1210	270	1730
	7	30	540	2220	1630	4390
S_8	2	6	570450	1980	22130	594560
	3	30	113550	26650	98460	238660
	5	30	240	1230	240	1710
	7	30	560	2270	1200	4030
S_9	3	6	143250	3120	240560	386930
	5	30	240	1240	240	1720
	7	30	550	2450	1530	4530
S_{10}	5	20	304800	36530	63212620	63553950
	7	30	590	2250	1230	4070

TABLE A.15. Ext Algebra Timings: Symmetric Groups

Group	Prime	n	Gen time	Spin Time	GRB Time	Total
M_{11}	2	30	8640	82600	2880	94120
	3	30	32890	13310	1372630	1418830
	5	100	560	20340	1490	22390
	11	100	1410	27110	7060	35580
M_{12}	2	12	16514760	15120	179400	16709280
	3	30	979680	42940	33315650	34338280
	5	100	790	18630	2420	21840
	11	100	3240	31300	12480	47020
J_1	2	30	497960	25340	1213210	1736510
	3	100	230	8920	600	9750
	5	100	250	9060	590	9900
	7	100	1870	32260	8830	42960
	11	40	2440	11810	23420	37670
	19	100	3030	40430	20480	63940
M_{22}	3	30	48900	11000	897740	957640
	5	40	310	2260	470	3040
	7	40	360	1740	360	2460
	11	40	450	2910	1230	4590
J_2	3	20	248690	6790	1654450	1909930
	5	24	295900	21860	1409550	1727310
	7	40	740	4170	1880	6790
M_{23}	3	30	147170	35220	896550	1078940
	5	40	210	2310	200	2720
	7	40	360	1890	370	2620
	11	40	440	2890	1230	4560
	23	100	5730	65390	106440	177560
HS	3	30	69930	13680	2253740	2337350
	5	8	1675200	6850	13624910	15306960
	7	30	680	1970	1500	4150
	11	30	940	2690	2500	6130
J_3	5	100	410	7860	800	9070
	17	100	4690	53260	38310	96260
	19	100	16610	109040	251970	377620
McL	2	8	2990780	3370	4747570	7741720
	5	14	7960800	26390	330778490	338765680
	7	100	640	12160	1580	14380
	11	100	3220	30150	11670	45040
M_{24}	5	100	1010	18280	2380	21670
	7	100	930	13360	2020	16310
	11	100	3920	55390	24940	84250
	23	100	6210	76630	93570	176410

Table A.16. Ext Algebra Timings: Sporadic Groups

Group	Prime	n	Gen time	Spin Time	GRB Time	Total
$L_2(7)$	2	40	44710	604520	9980	659210
	3	100	250	8870	620	9740
	7	100	1050	13430	2080	16560
$L_3(3)$	2	40	22540	250510	7840	280890
	3	30	774520	62010	17148860	17985390
	13	100	1110	13450	1820	16380
$L_2(8)$	2	20	153330	5590	1782330	1941250
	3	100	450	8390	740	9580
	7	100	300	9020	580	9900
$U_3(3)$	2	30	735220	12230	40600	788050
	3	40	188326	26269	5925881	6140476
	7	100	650	12460	1530	14640
$U_3(4)$	2	5	394620	1500	303940	700060
	3	100	250	8830	600	9680
	5	20	104450	3480	36470	144400
	13	100	730	12450	1550	14730
$U_3(5)$	2	30	2610	1300	2520	6430
	3	30	98640	22800	46440	167880
	5	8	881660	9680	680770	1572110
	7	100	650	12650	1510	14810
$U_4(2)$	2	10	2010410	3690	2324320	4338420
	3	10	570360	2840	2670000	3243200
	5	100	940	18630	2450	22020

TABLE A.17. Ext Algebra Timings: Classical Groups

Appendix B

Data Structures

Before we describe the implementations of our program in GAP we describe the data structures that are used. We do this first so that we can provide illustrative examples while describing our programs.

B.1 Basic Algebras

The main object that we will do our computations with is a basic algebra B. We are supplied with a faithful representation of the basic algebra in terms of matrices. The examples we include in chapter 5 are for the principal block of the basic algebra B. In GAP this is a record. It contains the following information:

- basicalg.field - the splitting field $\Bbbk$ for the basic algebra
- basicalg.group - the name of the original group G such that $\Bbbk G \cong_{\text{Morita}} B$.
- basicalg.generators - names of the vertices (idempotents) and edges (arrows) in the Ext-quiver.
- basicalg.npims - number of vertices in the Ext-quiver, also the number of PIMs in eBe.
- basicalg.pimnames - names of the PIMs in eBe given as strings such as "1a" to represent the PIM corresponding to the 1a in the representation of G. Note to see the original PIM names before condensation see Tom Hoffman's webpage math.arizona.edu/~hoffmant.
- basicalg.cartan - the Cartan matrix

- basicalg.matrices - This portion of the record contains as subrecords the names of the vertices (preceded by "pim") of the Ext-quiver. Contained in each of these subrecords are the compressed matrices which generate the basic algebra. Under the pim subrecord, after the matrices, is a subrecord spinning tree which is the spinning tree for this PIM in the basic algebra. The spinning tree is a data structure to keep track efficiently of the action of the generators. The spinning tree record is a list of records, where each record describes how to construct a basis vector as the image of the homomorphisms given. This construction was done as a spinning tree but has been reordered by the record perm. It contains a $\Bbbk$-basis for each of the PIMs. The words in the basis are ordered in terms of where they end, `basicalg.matrices.(pim).spinningtree[i].ende`. This makes it easier to look up information later just by consulting the Cartan matrix.

The following is an example of basic algebra record for the alternating group, S_4 for the field GF(2). Note in the record in GAP that the generators are strings such as "1a1a1", however we drop the quotations here. Also we record `0*Z(2)` as `0` and `Z(2)^0` as `1`.

```
gap> basicalg;
rec(group:=s4,generators:=[1a,2a,1a1a1,1a2a1,2a1a1,2a2a1],
 npims:=2, pimnames:=[1a,2a], cartan:=[[4,2],[2,3]], field:=GF(2),
 adjmat:=[[1,1],[1,1]], 1a:=rec(start:=1,ende:=1,name:=id1a),
 2a:=rec(start:=2,ende:=2,name:=id2a),
 1a1a1:=rec(start:=1,ende:=1,name:=1a1a1),
 1a2a1:=rec(start:=2,ende:=1,name:=1a2a1),
 2a1a1:=rec(start:=1,ende:=2,name:=2a1a1),
 2a2a1:=rec(start:=2,ende:=2,name:=2a2a1),
 matrices:=rec(
  pim1a:=rec(
    1:=[[1,0,0,0,0,0],[0,1,0,0,0,0],[0,0,1,0,0,0],
        [0,0,0,1,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0]],
    2:=[[0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0],
        [0,0,0,0,0,0],[0,0,0,0,1,0],[0,0,0,0,0,1]],
```

```
  3:=[[0,1,0,0,0,0],[0,0,0,0,0,0],[0,0,0,1,0,0],
      [0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0]],
  4:=[[0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0],
      [0,0,0,0,0,0],[0,0,1,0,0,0],[0,0,0,1,0,0]],
  5:=[[0,0,0,0,1,0],[0,0,0,0,0,1],[0,0,0,0,0,0],
      [0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0]],
  6:=[[0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0],
      [0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0]],
 perm:=[1,2,5,6,3,4],
 spinningtree:=[
  rec(ende:=1,name:=[],tree:=[]),
  rec(ende:=1,name:=[1a1a1],tree:=[1,3]),
  rec(ende:=1,name:=[2a1a1,1a2a1],tree:=[5,4]),
  rec(ende:=1,name:=[1a1a1,2a1a1,1a2a1],tree:=[6,4]),
  rec(ende:=2,name:=[2a1a1],tree:=[1,5]),
  rec(ende:=2,name:=[1a1a1,2a1a1],tree:=[2,5])]),
pim2a:=rec(
  1:=[[1,0,0,0,0],[0,1,0,0,0],[0,0,0,0,0],
      [0,0,0,0,0],[0,0,0,0,0]],
  2:=[[0,0,0,0,0],[0,0,0,0,0],[0,0,1,0,0],
      [0,0,0,1,0],[0,0,0,0,1]],
  3:=[[0,1,0,0,0],[0,0,0,0,0],[0,0,0,0,0],
      [0,0,0,0,0],[0,0,0,0,0]],
  4:=[[0,0,0,0,0],[0,0,0,0,0],[1,0,0,0,0],
      [0,0,0,0,0],[0,0,0,0,0]],
  5:=[[0,0,0,0,1],[0,0,0,0,1],[0,0,0,0,0],
      [0,0,0,0,0],[0,0,0,0,0]],
  6:=[[0,0,0,0,0],[0,0,0,0,0],[0,0,0,1,0],
      [0,0,0,0,1],[0,0,0,0,0]],
 perm:=[3,1,4,2,5],
 spinningtree:=[
  rec(ende:=1,name:=[1a2a1],tree:=[3,4]),
  rec(ende:=1,name:=[1a2a1,1a1a1],tree:=[1,3]),
  rec(ende:=2,name:=[],tree:=[]),
  rec(ende:=2,name:=[2a2a1],tree:=[3,6]),
  rec(ende:=2,name:=[1a2a1,2a1a1],tree:=[1,5])])))
```

From this record, we see we are looking at the field $\mathbb{F}_2$, a basic algebra with 6 generators, 2 PIMs of $\Bbbk$-dimension 6 and 5 respectively, and have 4 arrows. Thus the

Ext-quiver is given as:

$$\begin{tikzcd} 1a \arrow[loop left, "1a1a1"] \arrow[r, shift left, "2a1a1"] & 2a \arrow[l, shift left, "1a2a1"] \arrow[loop right, "2a2a1"] \end{tikzcd}$$

The arrows (maps between PIMs) are given as a string that first tells the terminus and then the origin of the arrow. For example, "1a2a1" is the map from PIM 2a to PIM 1a.

B.2 Gröbner Basis Information

This file is called `grbinf`. It is the result of running the program `GrbRecord`. It is a record of the following:

- grbinfo.groupname - the name of the group for our basic algebra $B \cong_{\text{Morita}} \Bbbk G$.
- grbinfo.field - the splitting field for B.
- grbinfo.generators - the generators of the basic algebra.
- grbinfo.pimnames - the PIM names of the basic algebra.
- grbinfo.npims - the number of PIMs for the basic algebra.
- grbinfo.cartan - the Cartan matrix of the basic algebra.
- grbinfo.adjmat - The adjacency matrix of the ext-quiver.
- grbinfo.nontips - This is the basis for the basic algebra extracted from B and reordered length lexicographically as the basic algebra is ordered PIM by PIM and by endings of words, not length lexicographically.
- grbinfo.tips - This is just a list of the tips.
- grbinfo.tipsrecords - This is the set of tips as well as a record of information for the start and end of the tips.

- grbinfo.minsharps - the set of MinSharps for the Gröbner basis $\mathcal{G}$.

For example, the grbinfo for S_4 in characteristic 2 is as follows:

```
gap> grbinfo; rec(groupname:=s4,field:=GF(2),
 generators:=[1a,2a,1a1a1,1a2a1,2a1a1,2a2a1],pimnames:=[1a,2a],
 1a:=rec(start:=1,ende:=1,name:=id1a),
 2a:=rec(start:=2,ende:=2,name:=id2a),
 1a1a1:=rec(start:=1,ende:=1,name:=1a1a1),
 1a2a1:=rec(start:=2,ende:=1,name:=1a2a1),
 2a1a1:=rec(start:=1,ende:=2,name:=2a1a1),
 2a2a1:=rec(start:=2,ende:=2,name:=2a2a1),
 npims:=2,cartan:=[[4,2],[2,3]],adjmat:=[[1,1],[1,1]],
 nontips:=[
  rec(name:=[1a],length:=0,ende:=1,start:=1,position:=1),
  rec(name:=[2a],length:=0,ende:=2,start:=2,position:=3),
  rec(name:=[1a1a1],length:=1,ende:=1,start:=1,position:=2),
  rec(name:=[2a1a1],length:=1,ende:=2,start:=1,position:=5),
  rec(name:=[1a2a1],length:=1,ende:=1,start:=2,position:=1),
  rec(name:=[2a2a1],length:=1,ende:=2,start:=2,position:=4),
  rec(name:=[2a1a1,1a2a1],length:=2,ende:=1,start:=1,position:=3),
  rec(name:=[1a1a1,2a1a1],length:=2,ende:=2,start:=1,position:=6),
  rec(name:=[1a2a1,1a1a1],length:=2,ende:=1,start:=2,position:=2),
  rec(name:=[1a2a1,2a1a1],length:=2,ende:=2,start:=2,position:=5),
  rec(name:=[1a1a1,2a1a1,1a2a1],length:=3,ende:=1,start:=1,
    position:=4)],
 nontiplist:=[[1a],[2a],[1a1a1],[2a1a1],[1a2a1],[2a2a1],[2a1a1,1a2a1],
  [1a1a1,2a1a1],[1a2a1,1a1a1],[1a2a1,2a1a1],[1a1a1,2a1a1,1a2a1]],
 tips:=[[1a1a1,1a1a1],[2a1a1,2a2a1],[2a2a1,1a2a1],[2a2a1,2a2a1],
  [2a1a1,1a2a1,1a1a1],[2a1a1,1a2a1,2a1a1],[1a2a1,1a1a1,2a1a1],
  [1a2a1,2a1a1,1a2a1]],
 tipsrecords:=[
  rec(name:=[1a1a1,1a1a1],basis:=[1a1a1],position:=2,
    generator:=1a1a1,ende:=1,start:=1),
  rec(name:=[2a1a1,2a2a1],basis:=[2a1a1],position:=5,
    generator:=2a2a1,ende:=2,start:=1),
  rec(name:=[2a2a1,1a2a1],basis:=[2a2a1],position:=4,
    generator:=1a2a1,ende:=1,start:=2),
  rec(name:=[2a2a1,2a2a1],basis:=[2a2a1],position:=4,
    generator:=2a2a1,ende:=2,start:=2),
  rec(name:=[2a1a1,1a2a1,1a1a1],basis:=[2a1a1,1a2a1],position:=3,
```

```
    generator:=1a1a1,ende:=1,start:=1),
  rec(name:=[2a1a1,1a2a1,2a1a1],basis:=[2a1a1,1a2a1],position:=3,
    generator:=2a1a1,ende:=2,start:=1),
  rec(name:=[1a2a1,1a1a1,2a1a1],basis:=[1a2a1,1a1a1],position:=2,
    generator:=2a1a1,ende:=2,start:=2),
  rec(name:=[1a2a1,2a1a1,1a2a1],basis:=[1a2a1,2a1a1],position:=5,
    generator:=1a2a1,ende:=1,start:=2)],
 minsharps:=[
  [[Z(2)^0,1a1a1,1a1a1]],
  [[Z(2)^0,2a1a1,2a2a1]],
  [[Z(2)^0,2a2a1,1a2a1]],
  [[Z(2)^0,2a2a1,2a2a1],[Z(2)^0,1a2a1,2a1a1]],
  [[Z(2)^0,2a1a1,1a2a1,1a1a1],[Z(2)^0,1a1a1,2a1a1,1a2a1]],
  [[Z(2)^0,2a1a1,1a2a1,2a1a1]],
  [[Z(2)^0,1a2a1,1a1a1,2a1a1],[Z(2)^0,1a2a1,2a1a1]],
  [[Z(2)^0,1a2a1,2a1a1,1a2a1]]])
```

The polynomials in minsharp correspond to the following set of 8 elements:

$$\begin{aligned}
&\{1a1a1 * 1a1a1, 2a1a1 * 2a2a1, 2a2a1 * 2a2a1 + 1a2a1 * 2a1a1, \\
&\quad 2a2a1 * 1a2a1 * 1a1a1 + 1a1a1 + 2a1a1 * 1a2a1, \\
&\quad 2a1a1 * 1a2a1 * 2a1a1, 1a2a1 * 1a1a1 * 2a1a1 + 1a2a1 * 2a1a1, \\
&\quad 1a2a1 * 2a1a1 * 1a2a1\}.
\end{aligned}$$

B.3 Anick Computation Record

Converted Gröbner Basis Information

To run our program, we need to compute normal forms and thus do division. Therefore in order to make a more efficient organization of the grbinfo data, we convert all of the grbinfo into more useful data. For example, we convert the entry `[Z(2)^0,2a2a1,2a2a1],[Z(2)^0,1a2a1,2a1a1]]`, i.e. 2a2a1*2a2a1+1a2a1*2a1a1, to `[[[4,4],[2,3]],[Z(2)^0,Z(2)^0]]`. We replace the generator name with its position in the arrows that are given in the list of generators in the basic algebra. We then write down the string of monomials as one entry in our list followed by the corresponding coefficients of the other. We will declare that `[[],[]]` is the zero in the field we are working in. If we would like a constant, then we have `[[],[Z(2)^0]]` = 1, for example.

We will also add to our grbinfo the information that comes from the fact that we are working in a path algebra. For example we will also add all incompatible arrows, that is start of second arrow does not equal end of first to the data for the minsharps. Therefore we now end up with what we label minsharpsplus:

```
[[[[1,1]],[Z(2)^0]], [[[3,4]],[Z(2)^0]], [[[4,2]],[Z(2)^0]],
 [[[4,4],[2,3]],[Z(2)^0,Z(2)^0]], [[[3,2,1],[1,3,2]],[Z(2)^0,Z(2)^0]],
 [[[3,2,3]],[Z(2)^0]], [[[2,1,3],[2,3]],[Z(2)^0,Z(2)^0]],
 [[[2,3,2]],[ Z(2)^0]], [[[1,2]],[Z(2)^0]], [[[1,4]],[Z(2)^0]],
 [[[2,2]],[Z(2)^0]], [[[2,4]],[Z(2)^0]], [[[3,1]],[Z(2)^0 ]],
 [[[3,3]],[Z(2)^0]], [[[4,1]],[Z(2)^0]], [[[4,3]],[Z(2)^0]] ].
```

Final Output in Resolution

While working on the computations in the Anick-Green resolution, we not only have a polynomial in the generators, but also a corresponding terminus which index the appropriate PIMs. For example in the resolution of S_4 we have the word $e_{\tau(c)}ba + e_{\tau(a)}cb$. We keep track of this as

```
[[[3],[[[2,1]],[Z(2)^0]]],[[1],[[[3,2]],[Z(2)^0]]]].
```

We combine all of the parts of our word according to the terminus. Thus as our example had two different termini, we had a list of size 2. Each part of this list was then of size 2 with the first part of the list being the terminus and the second part being the polynomial. To denote the zero word we will use `[[],[[],[]]]`. To denote just a terminus such as $\tau(a)$, we use `[[1],[[],[Z(2)^0]]]` for example.

The final resolution once computed will have the following records for each of steps in the resolution. The only information that is of ultimate importance is the matrix and the generators. The other information shows how we arrived at each step along the way and is used if we wish to compute the next step in the resolution. The data structure includes:

- p[n].Istar - The Gröbner basis for the one-point extension for that step in the resolution.

- p[n].T - This is a list of the tips in Istar including where it came from in previous level so we are able to compute higher overlaps.

- p[n].T2star - We compute the higher overlaps and it is the .name entry, and then we compute and store all of the information as in Theorem 3.18.

- p[n].redundancymat - Is a record of the generators after removing redundant ones, the redundant generators, the original matrix before reduction, and the reduced matrix.

- p[n].matrix - This is the map that we are interested in. It gives us ∂_n.

B.4 Cohomology and Ext Records

Projective Resolutions

After using our program `ProjectiveResolution`, a list of records is returned. Each of the records represent the projective modules in the resolution and also the map given as a list of images of the idempotents. The record for each of the steps is as follows:

- p[n].rowblocks - These are the idempotents for the PIMs in the image of the map ∂_n.

- p[n].generators - Gives a list of records which are images of the idempotents of P_n in the resolution.

  - p[n].generators[m].rowblocks - The idempotents for the PIMs in the image of the map ∂_n.

  - p[n].generators[m].blockvector - A partitioned vector that gives the image of the m^{th} idempotent of P_n in P_{n-1} under the map ∂_n.

- p[n].columnblocks - This is a list of the idempotents for the projective indecomposable modules in P_n. They are the domain of the map ∂_n.

```
gap> p[2];
 rec(rowblocks:=[1,2],generators:=[
   rec(blocks:=[1,2],blockvector:=[[0,1,0,0,0,0],[0,0,0,0,0]]),
   rec(blocks:=[1,2],blockvector:=[[0,0,1,0,0,0],[0,1,0,0,0]]),
   rec(blocks:=[1,2],blockvector:=[[0,0,0,0,0,0],[0,0,0,1,0]])],
  columnblocks:=[1,1,2]).
```

This means that in the minimal resolution we have $P_2 \to P_1$ is

$$e_{v_1}B \oplus e_{v_1}B \oplus e_{v_2}B \xrightarrow{\partial_2} e_{v_1}B \oplus e_{v_2}B$$

where the images of the idempotents are given by $(e_{v_1}, 0, 0) \mapsto (1a1a1, 0)$, $(0, e_{v_1}, 0) \mapsto (2a1a1 * 1a2a1, 1a2a1 * 1a1a1)$, and $(0, 0, e_{v_2}) \mapsto (0, 2a2a1)$.

Ext record

We first note that the cohomology record and Ext record are similar, thus we describe only the Ext record.

- ext.group - The original group for the basic algebra for which we are computing Ext.

- ext.field - The splitting field of the basic algebra.

- ext.generators - The generators for the Ext-algebra up to degree n.

- ext.pimnames - We label the idempotents from 1 to `basicalg.npims`.

- ext.n - This is the n such that we have $\bigoplus_{k=1}^{n} \bigoplus_{i,j} \mathrm{Ext}^k(S_i, S_j)$.

- ext.npims - Gives the number of pims.

- ext.basisforpims - This gives the basis written in terms of the generators.
- ext.actions - This gives the action of all of the standard basis elements on all of the generators.
- ext.grb - This gives a Gröbner basis $\mathcal{G}$ for the relations ideal up to degree n such that
$$\bigoplus_{i,j} \bigoplus_{k=1}^{n} \operatorname{Ext}_B^k (S_i, S_j) \cong \langle \text{Generators of } B \rangle / \langle \mathcal{G} \rangle.$$
- ext.homologydims - This is a $m \times m$ matrix where m is the number of pims. The $[i, j]$ entry of the matrix gives $\dim_{\Bbbk} \operatorname{Ext}^r_{\Bbbk G}(S_i, S_j)$ for $1 \leq r \leq n$ as a list.
- ext.repnames - This gives the original name of the idempotents (condensed PIMs) in the basic algebra.

Example We continue our example of S_4 by looking at the Ext-algebra up to degree $n = 2$.

```
rec(
 group:=S4,field:=GF(2), generators:=
   [[[1,1],[1,1]],[[1,2],[1,1]],[[2,1],[1,1]],[[2,2],[1,1]]],
 pimnames:=["1","2"],n:=2,npims:=2,
 basisforpims:=rec(
  1:=[
   rec(name:=[],start:=1,ende:=1,degree:=0),
   rec(name:=[[[1,1],[1,1]]],start:=1,ende:=1,
        degree:=1,vector:=[Z(2)^0]),
   rec(name:=[[[1,2],[1,1]]],start:=1,ende:=2,
    degree:=1,vector:=[Z(2)^0]),
   rec(name:=[[[1,1],[1,1]],[[1,1],[1,1]]],start:=1,ende:=1,
    degree:=2,vector:=[Z(2)^0,0*Z(2)]),
   rec(name:=[[[2,2],[1,1]],[[1,2],[1,1]]],start:=1,ende:=2,
    degree:=2,vector:=[Z(2)^0])],
  2:=[
   rec(name:=[],start:=2,ende:=2,degree:=0),
   rec(name:=[[[2,1],[1,1]]],start:=2,ende:=1,
    degree:=1,vector:=[Z(2)^0]),
```

```
  rec(name:=[[[2,2],[1,1]]],start:=2,ende:=2,
   degree:=1,vector:=[Z(2)^0]),
  rec(name:=[[[1,2],[1,1]],[[2,1],[1,1]]],start:=2,ende:=2,
   degree:=2,vector:=[Z(2)^0,Z(2)^0]),
  rec(name:=[[[2,1],[1,1]],[[2,2],[1,1]]],start:=2,ende:=1,
   degree:=2,vector:=[Z(2)^0]),
  rec(name:=[[[2,2],[1,1]],[[2,2],[1,1]]],start:=2,ende:=2,
   degree:=2,vector:=[0*Z(2),Z(2)^0])]),
 actions:=[[
  rec(start:=1,ende:=1,dims:=[1,2],
    cupspaces:=[[[Z(2)^0]],[[Z(2)^0,0*Z(2)],[0*Z(2),0*Z(2)]]],
    startnonzero:=1,v:=[(GF(2)^1),(GF(2)^2)],
    s:=[VectorSpace(GF(2),[[Z(2)^0]]),VectorSpace(GF(2),
    [[Z(2)^0,0*Z(2)],[0*Z(2),0*Z(2)]])],
    gens:=[
    rec(name:=[1,1],products:=[
       [rec(size:=1,result:=[[Z(2)^0]]),
        rec(size:=2,result:=[[Z(2)^0,0*Z(2)]])],
       [rec(size:=1,result:=[]),
        rec(size:=2,result:=[[0*Z(2)]])]],
       number:=1)]),
  rec(start:=1,ende:=2,dims:=[1,1],
    cupspaces:=[[[Z(2)^0]],[[0*Z(2)],[Z(2)^0]]],
    startnonzero:=1,v:=[(GF(2)^1),(GF(2)^1)],
    s:=[VectorSpace(GF(2),[[Z(2)^0]]),VectorSpace(GF(2),
[[0*Z(2)],[Z(2)^0]])],
    gens:=[
      rec(name:=[1,1],products:=[
       [rec(size:=1,result:=[]),
        rec(size:=2,result:=[[0*Z(2),0*Z(2)]])],
       [rec(size:=1,result:=[[Z(2)^0]]),
        rec(size:=2,result:=[[Z(2)^0]])]],
       number:=1)])],
 [
  rec(start:=2,ende:=1,dims:=[1,1],
    cupspaces:=[[[Z(2)^0]],[[0*Z(2)],[Z(2)^0]]],
    startnonzero:=1,v:=[(GF(2)^1),(GF(2)^1)],
    s:=[VectorSpace(GF(2),[[Z(2)^0]]),VectorSpace(GF(2),
[[0*Z(2)],[Z(2)^0]])],
    gens:=[
      rec(name:=[1,1],products:=[
```

```
        [rec(size:=1,result:=[[Z(2)^0]]),
         rec(size:=2,result:=[[0*Z(2)]])],
        [rec(size:=1,result:=[]),
         rec(size:=2,result:=[[Z(2)^0,Z(2)^0]])]],
     number:=1)]),
  rec(start:=2,ende:=2,dims:=[1,2],
    cupspaces:=[[[Z(2)^0]],[[Z(2)^0,Z(2)^0],[0*Z(2),Z(2)^0]]],
    startnonzero:=1,v:=[(GF(2)^1),(GF(2)^2)],
    s:=[VectorSpace(GF(2),[[Z(2)^0]]),VectorSpace(GF(2),
      [[Z(2)^0,Z(2)^0],[0*Z(2),Z(2)^0]])],
    gens:=[
      rec(name:=[1,1],products:=[
        [rec(size:=1,result:=[]),
         rec(size:=2,result:=[[Z(2)^0]])],
        [rec(size:=1,result:=[[Z(2)^0]]),
         rec(size:=2,result:=[[0*Z(2),Z(2)^0]])]],
     number:=1)])]],
  grb:=[
   [[[[[1,2],[1,1]],[[1,1],[1,1]]]],[Z(2)^0]],
   [[[[2,2],[1,1]],[[1,2],[1,1]]],
    [[[2,2],[1,1]],[[1,2],[1,1]]]],[Z(2)^0,Z(2)^0]],
   [[[[[1,1],[1,1]],[[2,1],[1,1]]]],[Z(2)^0]],
   [[[[[1,2],[1,1]],[[2,1],[1,1]]],
    [[[1,2],[1,1]],[[2,1],[1,1]]]],[Z(2)^0,Z(2)^0]],
   [[[[[2,1],[1,1]],[[2,2],[1,1]]],
    [[[2,1],[1,1]],[[2,2],[1,1]]]],[Z(2)^0,Z(2)^0]]],
homologydims:=[[[1,2],[1,1]],[[1,1],[1,2]]],
conjclassnames:=["1a","2a"])
```

Therefore to generate Ext up to degree 2, we have 4 generators. All of the generators are of degree 1. Therefore the Ext-quiver should be the same as the original quiver of the basic algebra. The generators are

`[[[1,1],[1,1]],[[1,2],[1,1]],[[2,1],[1,1]],[[2,2],[1,1]]].`

For example, the second generator is `[[1,2],[1,1]]` which means that it represents $\gamma \in \operatorname{Ext}(S_1, S_2)$ and that it is of degree 1 and comes from the 1$^{\text{st}}$ standard basis element. In general, `[[i,j],[k,l]]` means that it is a generator from $\operatorname{Ext}^k(S_i, S_j)$ of degree k and comes from the l^{th} standard basis element.

Index

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