Math 250A (Fall 2008) - Slope Fields Lab **SOLUTIONS**

Part I - Slope Fields

Consider the differential equation $\dfrac{dx}{dt} = at - bt^2$ where a and b are parameters.

(a) For which values of t are solution curves increasing? For which values of t are they decreasing? Explain your reasoning.

Answer: Solution curves [i.e., $x(t)$] will be increasing when the derivative is positive. Clearly, since we do not know what the values of a and b are, the answer will ultimately depend upon them. For $x(t)$ to be increasing, we must have $t(a - bt) > 0$. Remember that the inequality can flip when you divide both sides by a negative number, thus you need to be careful about the sign of both a and b. Ultimately, the sign of the b term is more important because it will have the dominant effect on the sign of the derivative for large values of t. If both a and b are positive, then increasing solutions correspond to $0 < t < a/b$. If both a and b are negative, then increasing solutions correspond to $t < 0$ or $t > a/b$. One can further flesh what happens when a and b have opposite sign or if either one is zero.

Note that another way to express your answer is graphically. If you plot $at - bt^2$ with respect to t, you will get a parabola (when $b \neq 0$, otherwise you get a straight line). The sign of b will determine whether the parabola is concave up ($b < 0$) or concave down. Points where the curve is above the t-axis correspond to t-values where solution curves are increasing. Note that the parabola passes through the t-axis at 0 and a/b.

(b) For which values of t are solution curves concave up? For which values of t are they concave down? Explain.

Answer: Concavity of the solution curves will depend upon second derivative with respect to t. For solutions to be concave up, we must have $a - 2bt > 0$. If b is positive, then this corresponds to $t < a/2b$. If b is negative, then this corresponds to $t > a/2b$.

(c) When is the right-hand-side even? When is it odd? What symmetries do you expect for the family of solution curves in each case? Explain your reasoning.

Answer: The right-hand side will be even [i.e., $f(-\alpha) = f(\alpha)$] when $a = 0$. It will be odd [i.e., $f(-\alpha) = -f(\alpha)$] when $b = 0$. When the differential equation is odd, all solution curves will be even. When the differential equation is even, the family of solution curves will be symmetric with respect to the origin. However, only one of the actual solution curves [where $x(t) = 0$] will actually be odd because the constant of integration will shift all other curves away from the origin such that $x(-t) \neq -x(t)$.

(d) Plot the slope field for $a = 1$ and $b = 1$. Use your answers to questions a–b above to find appropriate ranges for x and t. In particular, your plot should go far enough in the positive and negative x and t directions to include all of the salient features of the system. Paste the plot below.

<u>*Answer*</u>:

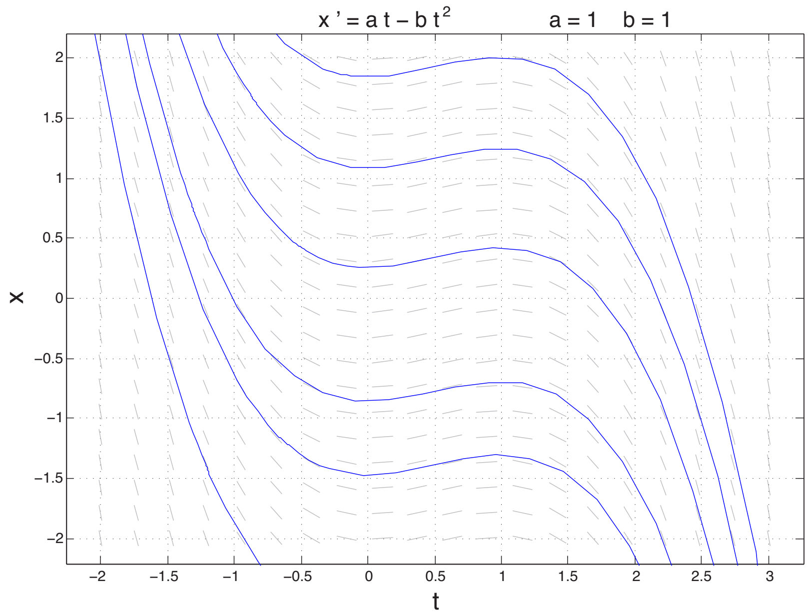

Figure 1: Problem I-d.

Solution curves are increasing for $0 < t < 1$ and decreasing for $t < 0$ or $t > 1$. Solution curves are concave up for $t < 1/2$ and concave down for $t > 1/2$.

(e) Solve the differential equation analytically. Your answer should depend on a and b.

<u>*Answer*</u>: This can be solved by direct integration. So we have

$$x(t) = \int \left(at - bt^2\right)\, dt = \frac{a}{2}t^2 - \frac{b}{3}t^3 + C$$

where C is an arbitrary constant of integration.

(f) Set $a = 1$ and $b = 1$. Find the solution that goes through the point $t = 0$, $x = 1$.

<u>*Answer*</u>: We have the initial condition $x(0) = 1$, so we can solve for C:

$$1 = \frac{1}{2}(0)^2 - \frac{1}{3}(0)^3 + C$$

Thus $C = 1$ and our solution is

$$x(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3 + 1$$

(g) Plot the solution you found above on the slope field (use a color other than blue) and paste the result below. Then, use DFIELD to plot the solution by clicking on the initial condition. Make sure that the numerical solution matches the solution you found.

Answer:

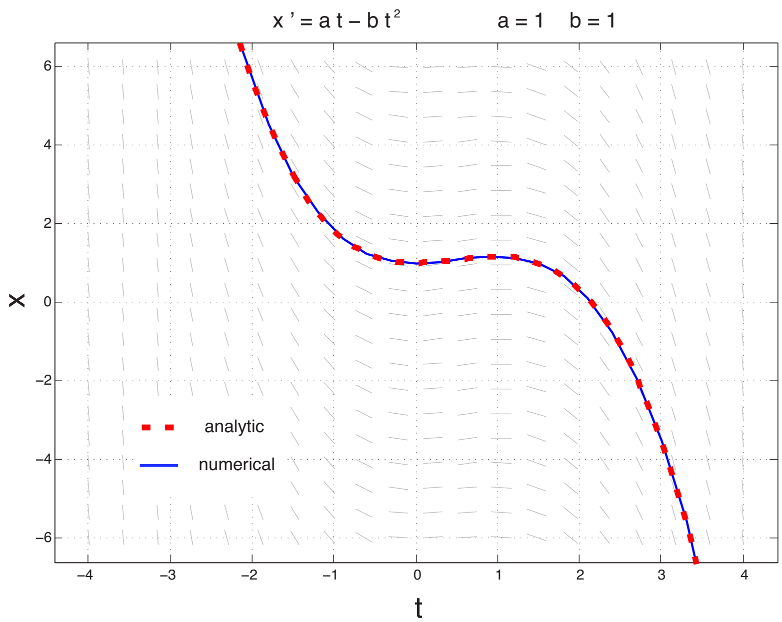

Figure 2: Problem I-g.

(h) Choose values of a and b for which the right-hand-side of the differential equation is odd and plot the corresponding slope field. Also use DFIELD to plot a few solution curves. Paste the output below. Does the family of solution curves have the expected symmetry? Why or why not?

Answer:
Let $a = 1$ and $b = 0$ to get the right-hand side odd.

In this case, all solution curves are even. See Figure 3.

(i) Repeat the above question when the right-hand-side of the differential equation is even. Show all your work.

Answer:
Let $a = 0$ and $b = 1$ to get the right-hand side even.

In this case, the family of solution curves is symmetric with respect to the origin. But only the solution passing through $x(0) = 0$ will truly be odd [see part (c)]. See Figure 4.

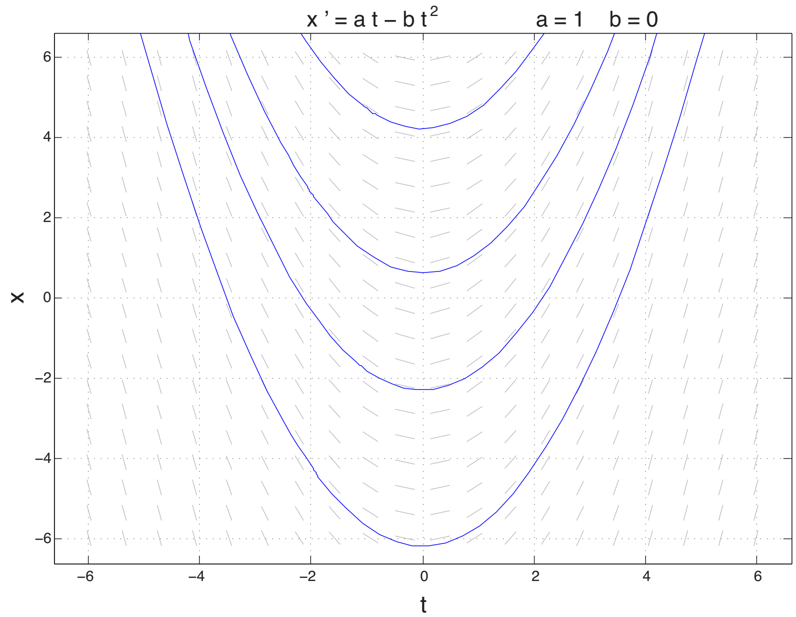

Figure 3: Problem I-h.

Part II - The Gompertz Equation

The evolution of the number of cells N in a growing tumor is often described by the Gompertz equation $\dfrac{dN}{dt} = -aN\ln(bN)$ where the parameters a and b are both positive.

(a) What is the sign of N ? Why?

<u>*Answer*</u>: Think about what N physically represents. Since in the context of this specific 'model' N represents the number of cells, we must have $N \geq 0$ since we can not have a negative number of cells.

As an aside, two other aspects are worth mentioning. In some models dealing with population sizes, the dependent variable can represent a population *difference* (e.g., the difference between the number of predator versus prey individuals). In those cases, it can potentially make sense to have a negative population size. Also, if N is a purely real number, one can not take the natural log of a natural number and thus N must be positive. However, if we are dealing with complex numbers (which commonly arise in various modeling contexts), it can be possible to take the logarithm when N is a negative number. To summarize, it is important to consider the specific context (and subsequent constraints) of the model you are dealing with.

(b) For which values of N are solution curves increasing? For which values of N are they decreasing? Explain your reasoning.

<u>*Answer*</u>: Solution curves [i.e., $N(t)$] will be increasing when the derivative is positive. Clearly, since we do not know what the values of a and b are, the answer will ultimately depend upon

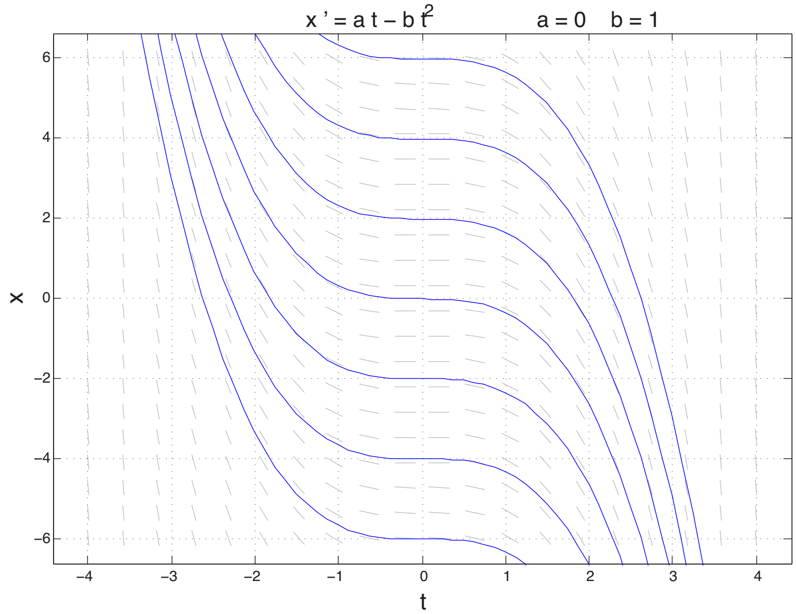

Figure 4: Problem I-i.

them. Since both a and b are positive, the right-hand side of the differential equation will be positive when $N > 0$ and if the argument of the logarithm is between zero and one. Thus solutions $N(t)$ will be increasing for $0 < N < 1/b$. For all other values of N (i.e., greater than $1/b$), $N(t)$ will be a decreasing function. Note that in contrast to Problem I, the answer here is independent of t.

(c) For which values of N are solution curves concave up? For which values of N are they concave down? Explain.

Answer: Concavity of the solution curves will depend upon the sign of the second derivative with respect to t. Note that the right-hand side of the differential equation is a function with respect to N, not t, thus extra care needs to be taken here. Using the chain and product rules, we have

$$\frac{d}{dt}\left[-aN\ln(bN)\right] = -a\ln(bN)\frac{dN}{dt} - a\frac{dN}{dt} = a^2N\ln(bN)\left[\ln(bN)+1\right]$$

The term a^2N will always be greater than zero (provided $N \neq 0$). The term $\ln(bN)$ will be positive when the argument is greater than one, or when $N > 1/b$, and negative otherwise. The term in square brackets will be positive when $\ln(bN) < -1$, or when $N < 1/be$ (where $e \approx 2.718...$) and negative otherwise. Thus solutions will be concave up when $0 < N < 1/eb$ or $N > 1/b$ and concave down when $1/eb < N < 1/b$.

(d) Plot the slope field for $a = 1$ and $b = 0.1$. Use your answers to questions a–c above to find appropriate ranges for N and t. In particular, your plot should go far enough in the relevant N and t directions to include all of the salient features of the system. Paste the plot below.

Answer:

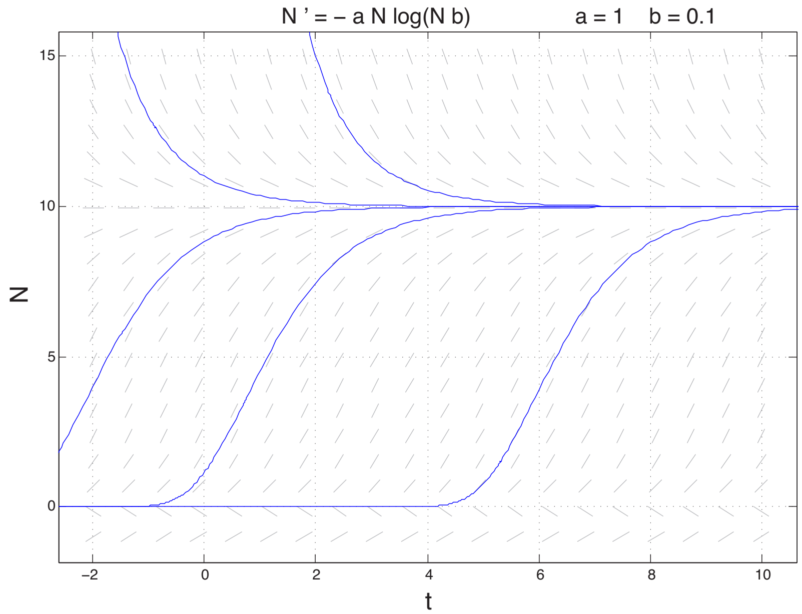

Figure 5: Problem II-d.

(e) Solve the differential equation analytically. Your answer (which involves two exponentials) should depend on a and b. Use your answer to find the limit of $N(t)$ as $t \to \pm\infty$. Explain what this means in terms of the model.

Answer: Similar to Part I, this equation can be solved by separation of variables (i.e., bring all terms having to do with N over to the left-hand side and all terms dealing with t to the right-hand side) and then integrate. However in this case, our integral will be with respect to N:

$$\int \frac{1}{N\ln(bN)}\,dN = -at + C$$

The integral can be solved by letting $u \equiv \ln(bN)$ such that $du = dN/N$. Making this substitution, our integral becomes

$$\int \frac{1}{u}\,du = \ln(u) + C' = -at + C$$

Combining constants, and exponentiating both sides, we have $u(t) = Ae^{-at}$. However, we need to re-substitute to get back an expression for N:

$$u(t) = \ln(bN) \Longrightarrow N(t) = \frac{\left(e^{e^{-at}}\right)^{A}}{b}$$

Thus our solutions takes the form of a recursive exponential. As $t \to \infty$, the term inside the parentheses will go to one (since $a > 0$ and the exponentiated exponential goes to 0) and thus N will approach $1/b$. In terms of the model, this means that the number of tumor cells will asymptotically approach a fixed value independent of what the initial condition is (provided $N \neq 0$, in which case the solution is singular and $N(t) = 0$ for all t).

(f) Set a = 1 and b = 0.1. Find the solution that goes through the point t = 8, N = 6.

Answer: We have the initial condition $N(8) = 6$. Plugging this in, we can solve for A:

$$6 = \frac{\left(e^{e^{-8}}\right)^A}{0.1} \implies A = \frac{\ln 0.6}{e^{-8}} \approx -1.52 \cdot 10^3$$

Thus our specific solution is

$$N(t) = 10\left(e^{e^{-t}}\right)^{e^8 \ln(0.6)} = 10 \cdot 0.6^{e^{8-t}}$$

(g) Plot the solution you found above on the slope field (use a color other than blue) and paste the result below. Then, use DFIELD to plot the solution. Make sure that the numerical solution matches the solution you found. Note that the natural logarithm in MATLAB is denoted by log (and the logarithm base 10 by log10).

Answer:

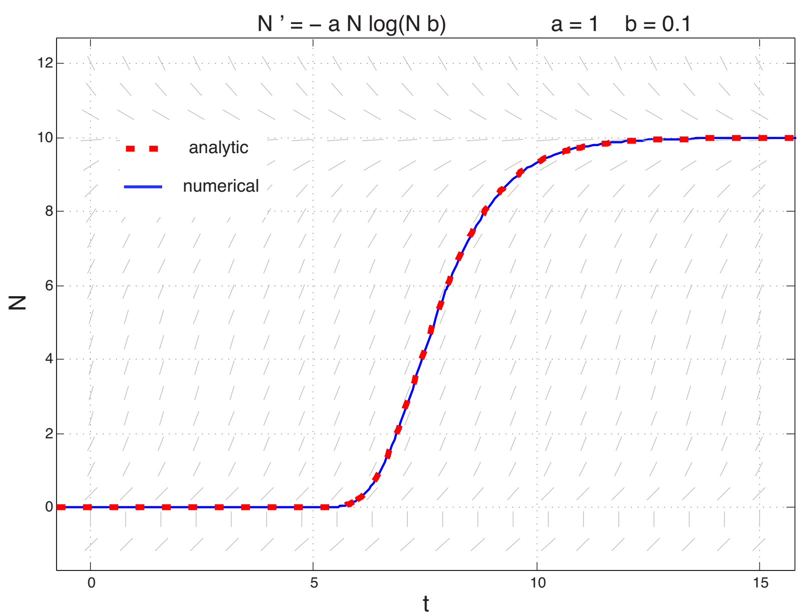

Figure 6: Problem II-g.

(h) Use DFIELD to explore how changing a affects the slope field and the solutions. What happens if you make the substitution $x = at$ in the above differential equation? Can you use this information to explain how the parameter a affects the slope field? Hint: think in terms of comparing the graph of a function $f(t)$ with the graph of the function $f(2t)$.

<u>*Answer*</u>:
The parameter a affects how quickly solutions approach the asymptotic limit. Larger values of a mean faster approach (i.e., solutions look steeper) while smaller a means a slower approach (i.e., solutions look more like a broad S-shape). If $a = 0$, solutions are just a constant (which makes sense seeing that the derivative is zero).
Let $x = at$, then $dx = a\,dt$. Plugging this substitution into the differential equation, we have

$$\frac{dN}{dx} = \frac{1}{a}\frac{dN}{dt} = -N\ln(bN)$$

Thus by making this substitution, we effectively get rid of the dependence upon a in the differential equation. The effect of this substitution relative to observation made using DFIELD makes sense because we can think of a as effectively scaling our time variable. Larger/smaller values of a means you are effectively compressing/expanding a unit of time and thus solutions appear to approach equilibrium faster/slower.

(i) Set $u = bN$ and find a differential equation for u. How can you use this information to explain how the parameter b affects the slope field?

<u>*Answer*</u>:
Let $u = bN$, then $du = b\,dN$. Plugging this substitution into the differential equation, we have

$$\frac{dN}{dt} = \frac{1}{b}\frac{du}{dt} = -\frac{au}{b}\ln(u) \Longrightarrow \frac{du}{dt} = -au\ln(u)$$

Thus we effectively get rid of the dependence upon b in the differential equation since making this substitution introducing removes the scaling (i.e., vertical shifting) of N. The asymptotic limit shifts as $1/b$, so it makes sense that the new limit becomes $u = bN = 1$ upon the substitution.

Note that we can also include both substitutions such that

$$\frac{du}{dx} = -u\ln(u),$$

thereby removing both a and b from the differential equation.

For both parts II-h and i, using substitutions of these types to effectively remove parameters from the differential equation can be very useful. This added degree of clarity comes because you remove unnecessary degrees of complexity when trying to understand the basic features of the differential equation. This notion of *scaling* occurs commonly in many types of problems.