Symmetries (continued)

$$y' = g(x)$$

Assume g is even i.e. $g(-x) = g(x)$

Assume $y(x)$ is a solution - Let

$u(x) = -y(-x)$.

$$\frac{du}{dx} = -\frac{d}{dx}y(-x) = -y'(-x)(-1)$$

$$= y'(-x) \underset{\substack{\uparrow \\ \text{from ode}}}{=} g(-x) \underset{\substack{\uparrow \\ g \text{ is even}}}{=} g(x)$$

So $\dfrac{du}{dx} = g(x)$.

Example: $y' = \dfrac{x^2+1}{x^2-1} = g(x)$

- $x^2-1=0$ for $x = \pm 1$ so g is not defined at $x = \pm 1$.

- g is continuous everywhere except at $x = \pm 1$.

  $\rightarrow$ Solutions exist on $(-\infty, -1)$, $(-1, 1)$ and $(1, +\infty)$

- Since $y' = \dfrac{x^2+1}{x^2-1} < 0$ on $(-1, 1)$, solution curves <u>decrease</u> on $(-1, 1)$.

- $$y'' = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) = \frac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2}$$

  $$= \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

  y'' is $\leqslant 0$ if $x \geqslant 0$ (where it exists)

  y'' is $\geqslant 0$ if $x \leqslant 0$ (")

- <u>Isocline</u>: a curve (or a collection of curves) such that

  $$\frac{x^2+1}{x^2-1} = m = \text{constant}$$

$$\frac{x^2+1}{x^2-1} = m \iff x^2+1 = m(x^2-1)$$

$$\iff x^2(m-1) = 1+m$$

$$\iff x^2 = \frac{m+1}{m-1}$$

$$\iff x = \pm\sqrt{\frac{m+1}{m-1}},$$

Sign of $\frac{m+1}{m-1}$ = sign of $(m+1)(m-1)$

Let $u(m) = (m+1)(m-1)$