Methods of integration II (continued)

1. Partial fractions (continued)

Recall that the general solution of

$$\frac{dy}{dx} = \frac{(y-1)^3}{y^4}$$

is (implicitly) given by

$$x + C = \frac{1}{2}y^2 + 3y + 6\ln|y-1| - \frac{4}{y-1} + \frac{-1}{2}\frac{1}{(y-1)^2}$$

Initial value problem 1 : $y(0)=2$

Substitute $x=0$, $y=2$ & find C

$$\cancel{0} + C = \frac{1}{2}4 + 6 + 0 - \frac{4}{1} + \frac{-1}{2\cdot 1}$$

$$= \frac{7}{2}$$

So the particular solution that goes through the point $(0,2)$ is given by

$$x + \frac{7}{2} = \frac{y^2}{2} + 3y + 6\ln|y-1| - \frac{4}{y-1} - \frac{1}{2(y-1)^2}$$

Initial value problem 2 : $y(0) = 1$

We cannot substitute this into the general solution.

Existence & uniqueness check out near $x=0$, $y=1$, so we know there is a unique solution going through $(0,1)$.

This must be a singular solution since it cannot be obtained from the general solution.

We know that $y=1$ is a singular solution & in fact this is the solution we are looking for.

So the solution that goes through $(0,1)$ is $\underline{y=1}$.

2. Trigonometric substitutions

Say you have to find the antiderivative of a function that contains a term in $\sqrt{1+x^2}$.

Question : how can I write $1+x^2$ as the square of something?

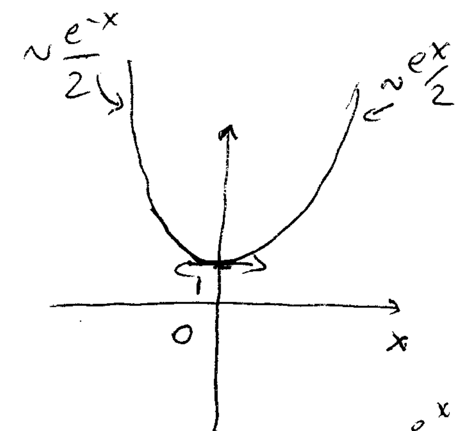
We know $\cos^2\theta + \sin^2\theta = 1$

So $$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

i.e. $1 + \tan^2\theta = \sec^2\theta$

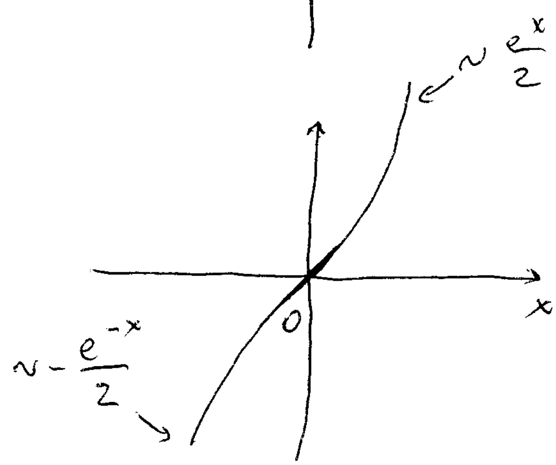
Set $x = \tan\theta$ so that $\sqrt{1+x^2}$ will become $\sqrt{\sec^2\theta} = |\sec\theta|$

Hyperbolic sine & cosine



$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\frac{d}{dx}\cosh(x) = \frac{e^x - e^{-x}}{2}$$
$$= \sinh(x)$$



$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\frac{d}{dx}\sinh(x) = \frac{e^x + e^{-x}}{2}$$
$$= \cosh(x)$$

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x}+2+e^{-2x}}{2^2} - \frac{e^{2x}-2+e^{-2x}}{4}$$

$$= \frac{1}{4}\left(e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}\right)$$

$$= 1$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}} = \frac{e^x-e^{-x}}{e^x+e^{-x}}$$

$$\frac{d}{dx}\tanh(x) = \frac{\cosh(x)\cdot\cosh(x)-\sinh(x)\cdot\sinh(x)}{\cosh^2(x)}$$

$$= \frac{\cosh^2(x)-\sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}$$

$$= 1 - \frac{\sinh^2(x)}{\cosh^2(x)} = 1-\tanh^2(x)$$

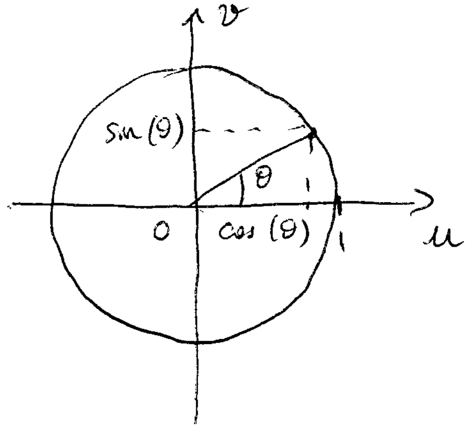
Application to trigonometric substitutions

1/ $\sqrt{a^2-x^2}$ Note that we must have $-a \leq x \leq a$

Set $x = a\sin(\theta)$ note that this implies $-a \leq x \leq a$

Then $a^2-x^2 = a^2 - a^2\sin^2(\theta) = a^2(1-\sin^2\theta)$
$= a^2\cos^2\theta$

So $\sqrt{a^2-x^2} = \sqrt{a^2\cos^2(\theta)} = a\,|\cos(\theta)|$



To have $x = a\sin(\theta)$ between $-a$ & a, we need to choose θ between $-\frac{\pi}{2}$ & $\frac{\pi}{2}$

In this case, $\cos(\theta) \geq 0$

So $a\,|\cos(\theta)| = a\cos(\theta)$

and $\sqrt{a^2-x^2} = a\cos(\theta)$

Table formula VI.28:

$$\int \frac{dx}{\sqrt{a^2-x^2}} \qquad \text{set } x = a\sin(\theta), \quad dx = a\cos(\theta)\,d\theta$$

$$= \int \frac{a\cos(\theta)\,d\theta}{a\cos(\theta)} = \int d\theta$$

$$= \theta + C = \arcsin\left(\frac{x}{a}\right) + C$$

2/ What if we have $\sqrt{x^2-a^2}$?
This implies that $x^2 \geq a^2$ i.e. $x \geq a$ or $x \leq -a$.

Without loss of generality (up to a $y=-x$ substitution), let us assume that $x \geq 0$

So we want $x \geq a$.

Let $x = a\cosh(\theta)$ so that $x \geq a$.

Then $\theta = \operatorname{arccosh}\left(\frac{x}{a}\right)$

Then $x^2 - a^2 = a^2\cosh^2(\theta) - a^2$

$$= a^2\left(\cosh^2(\theta) - 1\right)$$

$$= a\sinh^2(\theta)$$