Methods of integration II (continued)

Half-angle formulas (continued)

Let $t = \tan\left(\frac{\theta}{2}\right)$ $\quad dt = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)d\theta = \frac{1}{2}(1+t^2)\,d\theta$

$$\cos(\theta) = \frac{1-t^2}{1+t^2}$$

Obtain a similar formula for $\sin(\theta)$.

Recall $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

$\sin(2x) = 2\sin(x)\cos(x)$

$$\sin(\theta) = \frac{\sin(\theta)}{1} = \frac{\sin\left(2\frac{\theta}{2}\right)}{1}$$

$$= \frac{2\sin(\theta/2)\cos(\theta/2)}{\sin^2(\theta/2)+\cos^2(\theta/2)} = \frac{2\frac{\sin(\theta/2)}{\cos(\theta/2)}}{\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)}+1}$$

$$= \frac{2t}{1+t^2}$$

With the above change of variable, any integral of a sum, product or ratio of sines and cosines can be re-written as the integral of a rational function of t.

Example: Find $\int \frac{d\theta}{\sin(\theta)}$

Let $t = \tan\left(\frac{\theta}{2}\right)$ $\qquad dt = \frac{1}{2}(1+t^2)\,d\theta$

$$\sin(\theta) = \frac{2t}{1+t^2}$$

$$\int \frac{d\theta}{\sin(\theta)} = \int \frac{1+t^2}{2t} \frac{2\,dt}{1+t^2} = \int \frac{dt}{t}$$

$$= \ln|t| + C$$

$$= \ln\left|\tan\left(\frac{\theta}{2}\right)\right| + C$$