11/10/2008

Numerical approximations (continued)

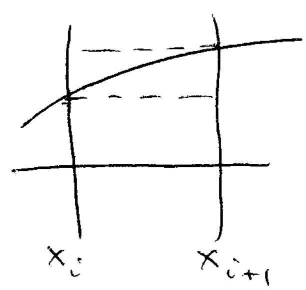
1. Approximation of definite integrals (continued)

Error: If the error is proportional to $\frac{1}{n^p}$

$$E = \frac{K}{n^p} \Rightarrow \ln|E| = \ln\left|\frac{K}{n^p}\right|$$
$$= \ln|K| - \ln|n^p|$$
$$= \ln|K| - p\ln|n|$$

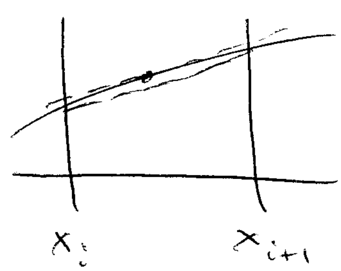
What is Simpson's rule?

LEFT, RIGHT



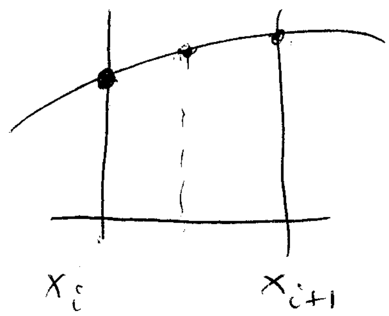
Approximate the function by a polynomial of degree 0

MID, TRAP



Approximate with a polynomial of degree 1.

Approximate with a polynomial of degree 2



Matches values of $f(x)$ at the left, middle & right points.

$$m = \frac{x_i + x_{i+1}}{2}$$

$$P_2(x) = f(x_i)\,\frac{x - x_{i+1}}{-x_{i+1} + x_i}\cdot\frac{x - m}{x_i - m} + f(m)\,\frac{x - x_{i+1}}{m - x_{i+1}}\cdot\frac{x - x_i}{m - x_i} + f(x_{i+1})\,\frac{x - x_i}{x_{i+1} - x_i}\cdot\frac{x - m}{x_{i+1} - m}$$

For Simpson's rule, replace f with $P_2(x)$ and approximate

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \quad \text{with} \quad \int_{x_i}^{x_{i+1}} P_2(x)\,dx$$

If you go through the calculation, you will find that

$$\int_{x_i}^{x_{i+1}} P_2(x)\,dx = f(x_i)\,\frac{\Delta x}{6} + f(m)\,\frac{2\Delta x}{3} + f(x_{i+1})\,\frac{\Delta x}{6}$$

$$= \frac{\Delta x}{3}\left(\frac{f(x_i) + f(x_{i+1})}{2} + 2\,f(m)\right)$$

2. Numerical integration of ode's

$$\frac{dy}{dx} = g(x, y)$$

$$y(x+h) - y(x) = \int_x^{x+h} g(t, y(t))\, dt$$

Approximate the integral using LEFT(1) so that

$$\int_x^{x+h} g(t, y(t))\, dt = h\, g(x, y(x))$$

Then, $y(x+h) - y(x) = h\, g(x, y(x))$

i.e. $y(x+h) = y(x) + h\, g(x, y(x))$

which Euler's method.

If we approximate the integral by MID(1) we have

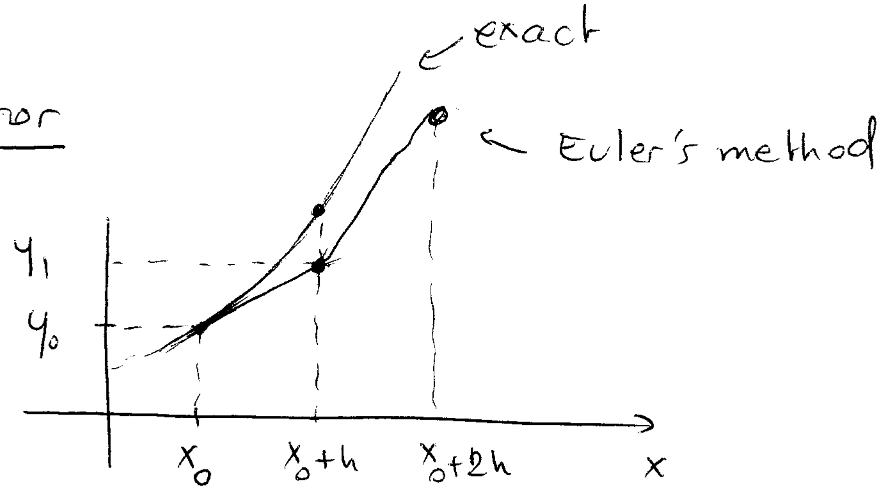
$$\int_x^{x+h} g(t, y(t))\, dt = h\, g\left(x + \frac{h}{2}, y\left(x + \frac{h}{2}\right)\right)$$

Since we don't know $y(x+\frac{h}{2})$, we replace it by its approximation using Euler's method:

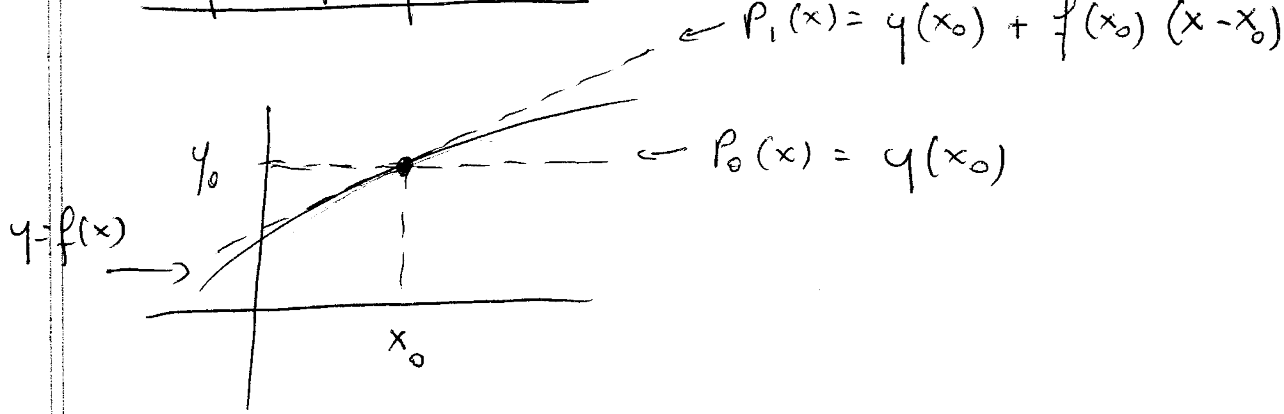
$$y\left(x + \frac{h}{2}\right) = y(x) + \frac{h}{2}\, g(x, y(x))$$

and $y(x+h) = y(x) + h\, g\left(x + \frac{h}{2}, y(x) + \frac{h}{2}\, g(x, y(x))\right)$

Error



3. Taylor polynomials



$$P_2(x) = y(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2}$$

$$P_2'(x) = f'(x_0) + f''(x_0)(x-x_0)$$

$$P_2''(x) = f''(x_0)$$

$$P_3(x) = P_2(x) + f'''(x_0)\frac{(x-x_0)^3}{3!}$$

$$P_n(x) = P_{n-1}(x) + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$$

such that $P_n^{(n)}(x_0) = f^{(n)}(x_0)$