Numerical approximations (continued)

More about the ξ

Recall that we had

$$f(x) = P_3(x) - \int_a^x \frac{(t-x)^3}{6} f^{(4)}(t)\, dt$$

We want to re-write this

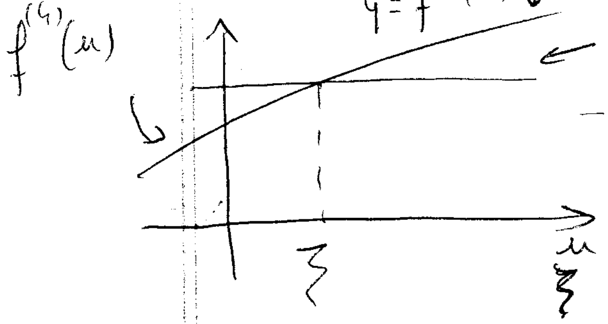
$$f(x) = P_3(x) + \frac{(x-a)^4}{4!} f^{(4)}(\xi)$$

$$= P_3(x) - \int_a^x \frac{(t-x)^3}{6} f^{(4)}(t)\, dt$$

So we want to find ξ such that

$$-\int_a^x \frac{(t-x)^3}{6} f^{(4)}(t)\, dt = \frac{(x-a)^4}{4!} f^{(4)}(\xi)$$

i.e. $$f^{(4)}(\xi) = -\frac{4!}{(x-a)^4} \int_a^x \frac{(t-x)^3}{6} f^{(4)}(t)\, dt = G(x)$$



Here, x is known i.e. $G(x)$ is just a number, that depends on x.

Instability of Euler's method

$$y' = \lambda y \qquad \frac{dy}{dx} = \lambda y \Rightarrow \frac{dy}{y} = \lambda dx$$

i.e. $\ln|y| = \lambda x + c$

i.e. $y = \underbrace{\pm e^{c}}_{K} e^{\lambda x} = K e^{\lambda x}$

Exact solution is $y = Ke^{\lambda x}$

If the initial condition is $y(0) = y_0$, then $y = y_0 e^{\lambda x}$.

Assume that $\lambda < 0$. $\lim_{x \to \infty} y(x) = 0$

Apply Euler's method to this:

$$y_{n+1} = y_n + h\, g(x_n, y_n) \qquad y' = g(x,y)$$

$$\frac{y_{n+1} - y_n}{\Delta x} \approx g(x_n, y_n)$$

i.e. $y_{n+1} = y_n + h \lambda y_n = (1 + \lambda h) y_n$

Say for $n=0$ $\quad y = y_0$

$n = 1 \quad y_1 = (1+\lambda h) y_0$

$n = 2 \quad y_2 = (1+\lambda h)^2 y_0$

$n \quad y_n = (1+\lambda h)^n y_0$

So if $|1+\lambda h| < 1$

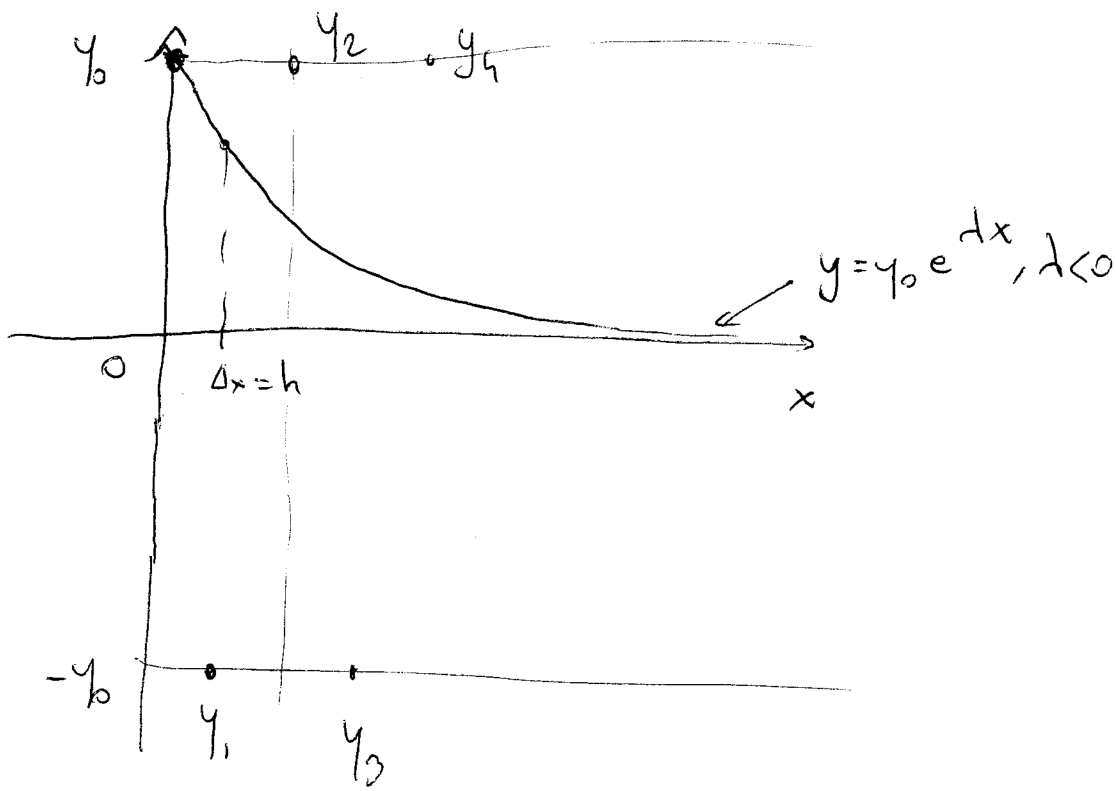
then $\lim_{n\to\infty} y_n = \lim_{n\to\infty} (1+\lambda h)^n y_0 = 0$

However, if $|1+\lambda h| > 1$

then $\lim_{n\to\infty} |y_n| = \lim_{n\to\infty} |1+\lambda h|^n |y_0| = +\infty$

If $1+\lambda h < 0$, then the sign of y_n alternates

If $|1+\lambda h| = 1$, then $|y_n| = |(1+\lambda h)^n| |y_0| = |y_0|$



Say $1+\lambda h = -1$. Then, $y_n = (-1)^n y_0$.

Another potential problem

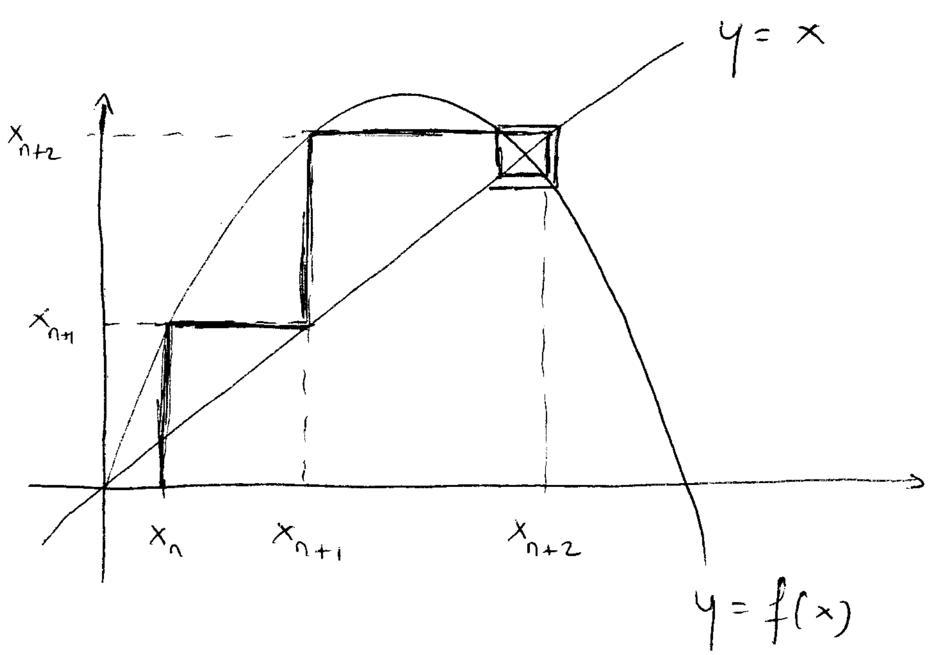
Nonlinear maps can be chaotic.

For a numerical method, say we have

$$y_{n+1} = F(y_n, x_n, h)$$

Example of the logistic map

$$y_{n+1} = a\, y_n (1 - y_n)$$



Cobweb diagram