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Equations of the form $y' = g(x,y)$ (continued)

3. Linear equations (continued)

Example: $xy' + y = \cos(x)$

You can see that the left-hand-side is $(xy)'$ so that the ode is

$$(xy)' = \cos(x)$$

i.e. $$xy = \int \cos(x)\,dx + c$$

$$= \sin(x) + c$$

i.e. $$y(x) = \frac{1}{x}\left(\sin(x) + c\right)$$

Apply the method of solution for linear equations (if you don't see that the l.h.s. is an exact derivative):

Re-write as $$y' + \frac{1}{x} y = \frac{\cos(x)}{x}$$

Method says: multiply by $\exp\left(\int \frac{dx}{x}\right)$
i.e. $\exp(\ln|x|) = |x| \leftarrow$ use x instead.

Example 2: $y' + \ln(x)\, y = \tan(x)$

Multiply by $\exp\left(\int \ln(x)\, dx\right)$

$$\int \ln(x)\, dx = x \ln(x) - \int x \frac{1}{x}\, dx$$
$$= x \ln(x) - x \quad (+C)$$

Multiply by $\exp(x \ln(x) - x)$

Then

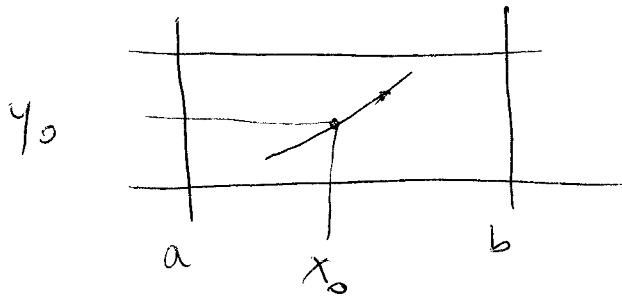
$$y' \exp(x \ln(x) - x) + \ln(x)\, y \exp(x \ln(x) - x)$$
$$= \tan(x) \exp(x \ln(x) - x)$$

i.e.
$$\frac{d}{dx}\left[y \exp(x \ln(x) - x)\right] = \tan(x) \exp(x \ln(x) - x)$$

i.e. $y \exp(x \ln(x) - x)$
$$= \int \tan(x) \exp(x \ln(x) - x)\, dx + C$$

i.e.
$$y(x) = \exp(-x \ln(x) + x) \int \tan(x) \exp(x \ln(x) - x)\, dx$$
$$+ C \exp(-x \ln(x) + x)$$

Existence & uniqueness



$$y' = q(x) - p(x)\,y$$

q, p continuous on $[a,b]$

$\Rightarrow$ solution exists & is unique on (a,b)

4. Bernoulli's equation

$$y' + p(x)\,y = q(x)\,y^n \qquad n \neq 1$$

Set $u = y^{1-n}$

$$\frac{du}{dx} = (1-n)\,y^{-n}\,y' = [q(x)\,y^n - p(x)\,y]\,y^{-n}(1-n)$$

$$= [q(x) - p(x)\,y^{1-n}](1-n) = [q(x) - p(x)\,u](1-n)$$

i.e. $u' + (1-n)p(x)\,u = q(x)(1-n)$, which is linear

Example: $y' = ay - y^3$