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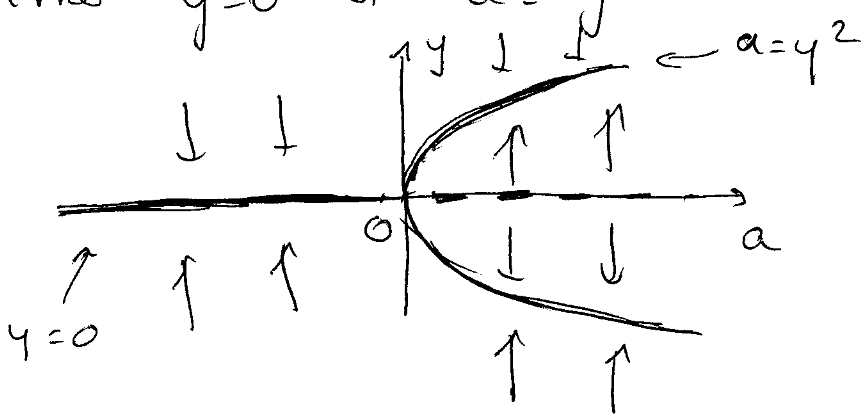
Equations of the form $y' = g(x,y)$ (continued)

4 Bernoulli equations (continued)

Example : $y' = ay - y^3$

Equilibria : $ay - y^3 = 0 \iff (a - y^2)\, y = 0$

either $y = 0$ or $a = y^2$



Now we can solve : $y' = ay - y^3$ $(n = 3)$

Let $u = y^{1-3}$
$= \frac{1}{y^2}$

Note : this implies $y \neq 0$

$$u' = \frac{-2}{y^3}\frac{dy}{dx} = -\frac{2}{y^3}\, y' = -\frac{2}{y^3}\left(ay - y^3\right)$$

$$= -2\frac{a}{y^2} + 2 = -2au + 2$$

This is linear in u : $u' + 2au = 2$

Solve this as a linear equation (as opposed to as a separable equation).

Multiply by e^{2ax} to get

$$e^{2ax}u' + 2au\,e^{2ax} = 2e^{2ax}$$

i.e. $$\left(u e^{2ax}\right)' = 2e^{2ax}$$

i.e. $$u e^{2ax} = \frac{1}{a}e^{2ax} + C \qquad a \neq 0$$

i.e. $$u = \frac{1}{a} + Ce^{-2ax} = \frac{1}{y^2}$$

i.e. $$y^2 = \frac{1}{\frac{1}{a} + Ce^{-2ax}} = \frac{a}{1 + Cae^{-2ax}}$$

Assume $y(0) = y_0$, then we can find C:

$$y_0^2 = \frac{a}{1+Ca} \Rightarrow 1 + Ca = \frac{a}{y_0^2} \qquad y_0 \neq 0$$

$$\Rightarrow Ca = \frac{a}{y_0^2} - 1$$

Then, $$y^2 = \frac{a}{1 + \left(\frac{a}{y_0^2} - 1\right)e^{-2ax}} = \frac{a y_0^2}{y_0^2 + (a - y_0^2)e^{-2ax}}$$

Solve for y:

$$y = \pm \sqrt{\frac{a\, y_0^2}{y_0^2 + (a - y_0^2)\, e^{-2ax}}}$$

But we know the solution is unique, so we can do better:

$$y = \pm \underbrace{\sqrt{\frac{a}{y_0^2 + (a - y_0^2)\, e^{-2ax}}}}_{\text{positive}} \; \underbrace{\sqrt{y_0^2}}_{\text{"}\pm y_0\text{"}}$$

$$= \pm y_0 \sqrt{\frac{a}{y_0^2 + (a - y_0^2)\, e^{-2ax}}}$$

Because y is continuous, we know that y must have the same sign as y_0, i.e.

$$y = y_0 \sqrt{\frac{a}{y_0^2 + (a - y_0^2)\, e^{-2ax}}}$$