Calculus and Differential Equations I

MATH 250 A

Methods of integration II

The method of partial fractions

- The purpose of the method of partial fractions is to find antiderivatives of rational functions, i.e. functions of the form $f(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials.
- The method involves three steps:
  1. If $d^o(P) \geq d^o(Q)$, first use long-division and re-write f as
  $$f(x) = N(x) + \frac{H(x)}{Q(x)}, \qquad d^o(H) < d^o(Q),$$
  where N and H are polynomials. Then, apply the method to the rational function $H(x)/Q(x)$.
  2. If $d^o(P) < d^o(Q)$, find the partial fraction decomposition of $P(x)/Q(x)$.
  3. Integrate each of the terms appearing in the partial fraction decomposition of f to obtain an antiderivative of f.

The method of partial fractions (continued)

- To do this, we need to be able to perform each of the steps separately. They are:
  1. Long-division of polynomials
  2. Partial fraction decomposition of $P(x)/Q(x)$ where $d^o(P) < d^o(Q)$
  3. Integration of terms that typically appear in a decomposition into partial fractions. Such terms are of the form
  $$\frac{A}{(x-a)^n} \qquad \text{and} \qquad \frac{Bx + C}{(x^2 + bx + c)^n},$$
  where $n \geq 1$ and $x^2 + bx + c$ is irreducible.
- Example for step 1: Divide x^3 by $x^2 + 3x + 2$.

Examples of application

1. We have already used partial fractions when solving the logistic equation.
2. Solve the following differential equation
$$\frac{dy}{dx} = \frac{(y+1)(y^2 - 2y + 3)}{y^2 + 5}.$$
3. Solve the differential equation
$$\frac{dy}{dx} = \frac{(y-1)^3}{y^4}$$
with the following initial conditions
   1. $y(0) = 2$
   2. $y(0) = 1$

Trigonometric substitutions

Trigonometric substitutions take advantage of known algebraic relationship between

- Sines, cosines, and tangents

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$\frac{d}{d\theta}\cos(\theta) = -\sin(\theta) \qquad \frac{d}{d\theta}\sin(\theta) = \cos(\theta)$$
$$\frac{d}{d\theta}\tan(\theta) = 1 + \tan^2(\theta) = \frac{1}{\cos^2(\theta)}$$

- Hyperbolic sines, cosines and tangents

$$\cosh^2(\theta) - \sinh^2(\theta) = 1$$
$$\frac{d}{d\theta}\cosh(\theta) = \sinh(\theta) \qquad \frac{d}{d\theta}\sinh(\theta) = \cosh(\theta)$$
$$\frac{d}{d\theta}\tanh(\theta) = 1 - \tanh^2(\theta) = \frac{1}{\cosh^2(\theta)}$$

Trigonometric substitutions (continued)

- For integrands that involve $\sqrt{a^2 - x^2}$, $a > 0$, note that $|x| \leq a$, and try the substitution $x = a\sin(\theta)$.
- Since the integrand will involve $\sqrt{\cos^2(\theta)}$ and the dx will be given by $dx = a\cos(\theta)\,d\theta$, one can expect to be able to simplify the integral after such a substitution.
- Example: Show that $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$.
- Similarly, for integrands that involve $\sqrt{x^2 - a^2}$, $a > 0$, one can change variables so that $x > 0$ and then try $x = a\cosh(\theta)$ since $x^2 \geq a^2$.
- Examples: Show that $\displaystyle\int \sqrt{x^2 - a^2}\,dx$ can be written as $a^2 \displaystyle\int \sinh^2(\theta)\,d\theta$ after a substitution.

Half-angle substitutions

- Half-angle substitutions are useful to find antiderivatives of products and/or ratios of sines and cosines.
- Indeed, let $t = \tan(\theta/2)$. Then,

$$\cos(\theta) = \frac{1 - t^2}{1 + t^2}, \qquad \sin(\theta) = \frac{2t}{1 + t^2}, \qquad dt = \frac{1}{2}(1 + t^2)\,d\theta.$$

- A product or ratio of sines and cosines will thus be transformed into a rational function of t, which we know how to integrate (using partial fractions).
- Example: Show that $\displaystyle\int \frac{d\theta}{\sin(\theta)} = \ln\left|\tan\left(\frac{\theta}{2}\right)\right| + C$.

Partial fraction decomposition

To decompose the rational function $\dfrac{P(x)}{Q(x)}$ where $d^{o}(P) < d^{o}(Q)$, into partial fractions, proceed as follows.

1. Factor the denominator $Q(x)$ into terms of the form $(x - a)^n$ and $(x^2 + bx + c)^n$, where $n \geq 1$ and $x^2 + bx + c$ is irreducible.
2. For each factor of the form $(x - a)^n$, the partial fraction decomposition of $P(x)/Q(x)$ will include terms of the form

$$\frac{A_1}{x - a}, \frac{A_2}{(x - a)^2}, \ldots, \frac{A_j}{(x - a)^j}, \ldots, \frac{A_n}{(x - a)^n}.$$

3. To find A_n, multiply by $(x - a)^n$ and set $x = a$ into the resulting equation.
4. To find the A_j's, $j \neq n$, multiply by $(x - a)^n$, and substitute in appropriate values of x.

Partial fraction decomposition (continued)

5. For each factor of the form $(x^2+bx+c)^n$, the partial fraction decomposition of $P(x)/Q(x)$ will include terms of the form

$$\frac{B_1x+C_1}{x^2+bx+c},\ \cdots,\ \frac{B_jx+C_j}{(x^2+bx+c)^j},\ \cdots,\ \frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

6. To find the B_j's and C_j's, multiply by $(x^2+bx+c)^n$, expand, and equate the coefficients of the various powers of x in both sides of the resulting equation.

Example: Find the partial fraction decomposition of

$$f(x)=\frac{x^2+5}{(x+1)(x^2-2x+3)}.$$

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Integration of a partial fraction decomposition

Typical terms in a partial fraction decomposition are of the form

$$\frac{A}{(x-a)^n} \quad \text{and} \quad \frac{Bx+C}{(x^2+bx+c)^n}.$$

1. Terms of the form $\dfrac{A}{(x-a)^n}$, $n\geq 1$
   - If $n=1$, then
     $$\int \frac{A}{x-a}\,dx=\ln(|x-a|)+C.$$
   - If $n>1$, then
     $$\int \frac{A}{(x-a)^n}\,dx=\frac{-A}{n-1}\frac{1}{(x-a)^{n-1}}+C.$$

Integration of a partial fraction decomposition (continued)

2. Terms of the form $\dfrac{Bx+C}{(x^2+bx+c)^n}$, $n\geq 1$
   1. Compare the numerator to the derivative of x^2+bx+c.
      $$\begin{aligned}\int \frac{Bx+C}{(x^2+bx+c)^n}\,dx &= \int \frac{\frac{B}{2}(2x+b)-\frac{bB}{2}+C}{(x^2+bx+c)^n}\,dx\\ &= \frac{B}{2}\int\frac{du}{u^n}+D\int\frac{dx}{(x^2+bx+c)^n},\end{aligned}$$
      where $u=x^2+bx+c$ and $D=C-\dfrac{bB}{2}$.
   2. Thus, we can integrate provided we know how to find an antiderivative of $1/(x^2+bx+c)^n$.
   3. Note that since x^2+bx+c is irreducible, one can write $x^2+bx+c=\left(x+\dfrac{b}{2}\right)^2+d^2$, where $d^2=c-\dfrac{b^2}{4}$.

Integration of a partial fraction decomposition (continued)

3. To integrate $\dfrac{1}{\left(\left(x+\frac{b}{2}\right)^2+d^2\right)^n}$, let $u=\dfrac{x}{d}+\dfrac{b}{2d}$. Then,

$$\int\frac{dx}{\left(\left(x+\frac{b}{2}\right)^2+d^2\right)^n}=\frac{1}{d^{2n-1}}\int\frac{du}{(u^2+1)^n}.$$

   - If $n=1$, then
     $$\int\frac{dx}{\left(x+\frac{b}{2}\right)^2+d^2}=\frac{1}{d}\int\frac{du}{(u^2+1)}=\frac{1}{d}\arctan\left(\frac{x}{d}+\frac{b}{2d}\right)+C$$
   - If $n>1$, let $\theta=\arctan(u)$. Then, $d\theta=\dfrac{du}{1+u^2}$ and
     $$\int\frac{du}{(u^2+1)^n}=\int\frac{d\theta}{(1+\tan^2(\theta))^{n-1}}=\int\cos^{2n-2}(\theta)\,d\theta.$$
     Alternatively, integrate by parts and find a recursive formula.

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