

Calculus and Differential Equations I

MATH 250 A

Introduction to differential equations

Introduction to differential equations

Calculus and Differential Equations I

A simple differential equation

- Is there a function which is equal to its derivative?
 - 1 Yes
 - 2 No
- Is such a function unique?
 - 1 Yes
 - 2 No
- The equation $\frac{dy}{dx} = y$ is an example of an **ordinary differential equation**.
- The **independent variable** is x and the **dependent variable** is y .
- The above equation is a **first order**, **autonomous** and **linear** differential equation.

Introduction to differential equations

Calculus and Differential Equations I

Solutions of a differential equation

- An **explicit solution** of an ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

is a function $y(x)$ such that when substituted into the differential equation, both sides are found to be identical.

- In general, a given differential equation will have a **family of solutions**, involving **one or more parameters**.
- Applying **initial** or **boundary** conditions often leads to the **selection** of one of these solutions.
- We will now turn to two important questions: **what are differential equations used for**, and **how do we study them**?
- To address the first questions, we now look at **examples** of differential equations or systems thereof.

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Calculus and Differential Equations I

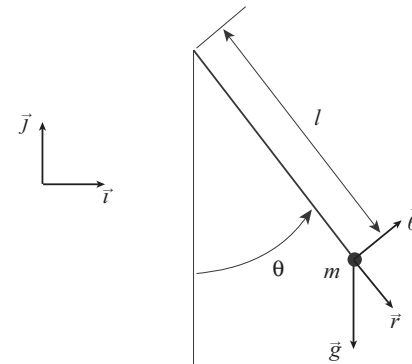
The nonlinear pendulum

The equation of motion for the **nonlinear pendulum** is given by

$$m l \frac{d^2\theta}{dt^2} = -m g \sin(\theta) - c l \frac{d\theta}{dt},$$

where

- θ and t are **variables**.
- m , l , g and c are **parameters**.
- Most of these quantities are defined on the figure, except c , which measures friction.



Sketch of a point-mass pendulum

Introduction to differential equations

Calculus and Differential Equations I

The RLC circuit

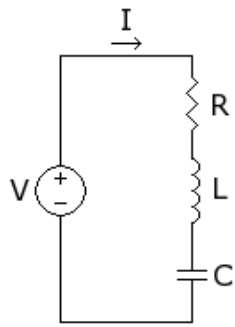


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The series **RLC circuit** consists of a **resistor** of resistance R , an **inductor** of inductance L , a **capacitor** of capacitance C , and a power source of voltage $V(t)$.

The charge q across the capacitor satisfies the differential equation

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = V(t),$$

and the current in the circuit is given

$$\text{by } I(t) = \frac{dq}{dt}.$$

The classic SIR model



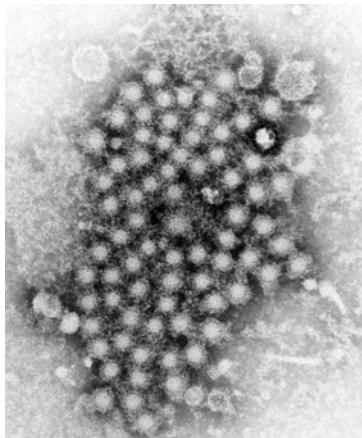
Penned goats in a village within a region investigated for a Rift Valley fever outbreak in Saudi Arabia. Picture # 8362, Public Health Image Library

The **classic SIR model** reads

$$\begin{aligned} \frac{dS}{dt} &= -\alpha S \frac{I}{N}, \\ \frac{dI}{dt} &= \alpha S \frac{I}{N} - \beta I, \\ \frac{dR}{dt} &= \beta I, \end{aligned}$$

where S , I , and R represent the numbers of **susceptible**, **infectious**, and **recovered** (or **removed**) individuals, in a population of size N . The parameter α measures the average number of **positive contacts** per susceptible per unit of time, and β measures the **rate at which individuals recover**.

Viral infections



Transmission electron micrograph showing hepatitis B virions of an unknown strain. Picture # 8153, Public Health Image Library

The the dynamics of a **viral infection**, such as hepatitis B or C, may be described by the following model (M.A. Nowak *et al.*, Proc. Natl. Acad. Sci. USA **93**, 4398-4402 (1996)).

$$\begin{aligned} \frac{dX}{dt} &= \lambda - \delta X - b V X \\ \frac{dY}{dt} &= b V X - a Y \\ \frac{dV}{dt} &= k Y - \kappa V \end{aligned}$$

The variable X represents the number of **uninfected cells**, Y is the number of **infected cells**, and V is the **viral load** (or number of free virions in the body).

How do we study differential equations?

- Sometimes, we can **solve** a differential equation. In this class (MATH 250 A & B), we will learn how to solve **first and second order linear equations** and **systems of first order linear equations**, as well as **some first order nonlinear equations**.
- If **initial conditions are known**, one can solve a differential equation (or a system of differential equations) **numerically**. We will learn a simple **numerical method** to solve a differential equation and also use more advanced algorithms in MATLAB.
- Before trying to solve a differential equation, or launching into a numerical exploration of its properties, one needs to know **whether solutions exist** and if so, **whether they are unique**. We will see theorems that guarantee existence and uniqueness of solutions to differential equations.

How do we study differential equations? (continued)

- In many situations, especially when one deals with **nonlinear** differential equations, one **cannot find explicit solutions**.
- In this case, one can nevertheless **understand the dynamics** of a differential equations by looking at **special solutions** and at their **stability**.
- The **qualitative theory of dynamical systems** (discussed in MATH 454) provides a way of understanding the behavior of a system of differential equations, as well as the **bifurcations** that occur when one or more **parameters are changed**. We will briefly address some of these issues.
- **Partial differential equations** (discussed in MATH 322, MATH 422, and MATH 456) are differential equations describing the dynamics of systems with **two or more independent variables**.

What we will do next

- We will start with the **simplest type of differential equations**,
 $\frac{dy}{dx} = g(x)$.
 - **Chapter 1** of *Differential Equations* book. **Reading assignment:** Sections 1.1, 1.2, and 1.3.
 - Solving such differential equations involves integration, so we will introduce various **methods of integration** discussed in the *Calculus* book.
- We will then consider **autonomous differential equations** of the form $\frac{dy}{dx} = g(y)$.
 - **Chapter 2** of the *Differential Equations* book.
 - Ideas of **stability** and **bifurcations**.
 - One more technique of integration: **partial fractions**.

What we will do next (continued)

- We will then turn to **general first order differential equations**,
 $\frac{dy}{dx} = g(x, y)$
 - **Chapter 3** of the *Differential Equations* book.
 - Graphical analysis, **symmetries**, **scalings**, and **numerical solutions**.
 - **Existence and uniqueness** of solutions.
- Finally, we will look at various **methods of solution** for first order differential equations
 - **Chapters 4 and 5** of the *Differential Equations* book.
 - **Separation of variables** and equations with **homogeneous coefficients**.
 - First order **linear** differential equations and **Bernoulli's equation**.