Calculus and Differential Equations I

MATH 250 A

Modeling with differential equations

Objects in motion

- Newton's law: for an object moving in one dimension

$$F = m\gamma = m\frac{dv}{dt},$$

where F is the sum of forces applied along the positive x direction, m is the mass of the object, x is the position of its center of mass, and $v = dx/dt$ is its velocity.

- If the only force is gravity, then $F = -mg$ if x points upward in the vertical direction. In this case, we have

$$\frac{dv}{dt} = -g,$$

which is solved by direct integration.

Objects in motion (continued)

- In the presence of gravity and friction, we typically have
  - $F = -m\,g - c\,v$, $c > 0$, if the object is moving slowly.
  - $F = -m\,g + c\,v^2$, $c > 0$, if $|v| >> 1$, $v < 0$.
  - It is possible to have other types of friction forces, especially in the case of solid friction.
- If we restrict ourselves to the above examples, we have
  - $\frac{dv}{dt} = -g - \frac{c}{m}v$, which is linear in v.
  - $\frac{dv}{dt} = -g + \frac{c}{m}v^2$, which is separable.
- For a spring-mass system, we have $F = -k(x - x_0)$, $k > 0$. Then, $m\frac{d^2x}{dt^2} = -k(x - x_0)$, which is a second order, linear equation.

Mixture problems

- These problems typically involve a fluid, of volume $V(t)$, in which a substance is dissolved. The goal is to find the amount $A(t)$ or the concentration $C(t) = A(t)/V(t)$ of the substance in the fluid.
- The general way of addressing such a problem is to write a balance equation for the amount $A(t)$ of the substance in the fluid,

$$\frac{dA}{dt} = \text{input rate} - \text{output rate}$$

- Example (#5 page 207): Take a 200-gallon container filled with pure water. Add a salt concentration with 3 pounds of salt per gallon, at a rate of 4 gallons per minute. At the same time, drain the container at a rate of 5 gallons per minute. Find the amount of salt in the container as a function of time.

Cooling and heating

- Newton's law of cooling and heating says that the rate of change of the temperature T of an object is a linear function of the difference between T and the ambient temperature T_0:

$$\frac{dT}{dt} = -k(T - T_0), \qquad k > 0.$$

- This equation can be solved as a linear equation, or as a separable equation, to find

$$T(t) = T_0 + \kappa \exp(-kt),$$

where κ is an arbitrary constant.

- As expected, $T \to T_0$, as $t \to +\infty$.

Compounding interest

- If money in a bank account is compounded continuously at a rate of r percents per year, then in the absence of deposits or withdrawals, we have

$$\frac{dM}{dt} = \frac{r}{100} M,$$

where M is the account balance and t is time measured in years.

- The above equation describes the exponential growth of M.
- After one year, the amount of money in the account is given by

$$M(1) = \exp(r/100)\, M(0).$$

- The annual interest rate is therefore larger than $r/100$, since

$$APY = \exp(r/100) - 1.$$

Population dynamics

- If N is the population density of a region, then one can write

$$\frac{dN}{dt} = bN - dN + \text{immigration} - \text{emigration},$$

assuming that resources are not limited.

- In the above equation, b is the birth rate, and d is the death rate of the population. The growth rate r of the population is given by

$$r = b - d.$$

- If immigration and emigration are given functions of t, then the above equation is linear in N.

Population dynamics (continued)

- If a population is growing exponentially at rate $r > 0$, we can define its doubling time

$$T_d = \frac{\ln(2)}{r}.$$

- Note the analogy with the half-life of a substance decaying exponentially at rate $r < 0$,

$$T_{1/2} = \frac{-\ln(2)}{r} = \frac{\ln(2)}{|r|}.$$

- If resources are limited, one can expect that r will depend on N. With $r = \alpha - \beta N$, $\alpha > 0$, $\beta > 0$, and in the absence of immigration or emigration, we have logistic growth

$$\frac{dN}{dt} = \alpha N - \beta N^2.$$

Chemical reactions

- For a chemical reaction of the form $A + B \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C$, the law of mass action says that

$$\frac{d[C]}{dt} = k_1[A]\,[B] - k_2[C],$$

where $[X]$ is the concentration of chemical X and k_1 and k_2 are the forward and backward rate constants respectively.

- For an autocatalytic reaction of chemical X, one may have

$$\frac{d[X]}{dt} = k_1 a[X] - k_2[X]^2,$$

where a, k_1, and k_2 are constants. This is again the logistic equation.