Calculus and Differential Equations I

MATH 250 A

Differential equations of the form $y' = g(x)$

Formal solutions

- The differential equation $y' = g(x)$, where g is continuous, may formally be solved by integration, so that

$$y(x) = \int g(x)\ dx + C.$$

- Fundamental theorem of Calculus: if f is a continuous function on $[a, b]$ and if $f(x) = F'(x)$, then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

- Second fundamental theorem of Calculus: if f is continuous on $[a, b]$ and if $x \in [a, b]$, then

$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x).$$

Formal solutions (continued)

- We can therefore write the general solution to $y' = g(x)$ as

$$y(x) = \int_a^x g(t)\ dt + C.$$

- If we can find an antiderivative of $g(x)$, then we have an explicit solution.
- Given a continuous function g, can we always find an explicit expression for an antiderivative of g?
  1. Yes
  2. No
- There exists functions which we do not know how to integrate. An example is the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\ dt.$$

Formal solution (continued)

- Even if we do not know how to integrate g, we may still be able to say something about the behavior of $y(x)$ by looking at properties of the antiderivative of g.
- As $x \to \infty$, the integral in the expression for y becomes an improper integral of the form $\int_a^\infty g(t)\ dt$. We will study improper integrals in the second semester of this course.
- An initial or boundary condition of the form $y(x_0) = y_0$ allows us to pick a particular solution from the one-parameter family of solutions:

$$y(x) = \int_{x_0}^x g(t)\ dt + y_0.$$

- A solution curve is the graph of a particular solution $y(x)$.

Existence and uniqueness of solutions

- True/False: All solution curves of $y' = g(x)$ may be obtained by vertical translation of one of them.
  1. True
  2. False
- Since we assume that g is continuous, we know that solutions exist. Are they unique?
- If there is a unique solution for any initial condition, is it possible for solution curves to cross (or meet) at a point?
  1. Yes
  2. No
- If at some point g becomes singular, e.g. if it is undefined or it stops being continuous, the argument for existence, which assumes that g is continuous, will fail.
- Nevertheless, in some cases it will be possible to patch solutions of a differential equation found in adjacent intervals.

Qualitative properties of solutions

- Since we know y', and assuming that g is smooth, we know all of the derivatives of y.
- As a consequence, we know when the solution y is increasing or decreasing, and we know the concavity of the graph of y.
- We can use symmetries of the equation to relate a solution to another solution. For instance, if $y(x)$ is a solution and $g(x)$ is odd, then $u(x) = y(-x)$ is also a solution.
- If $y(x)$ is a solution and $g(x)$ is even, then which of the solutions below is also a solution?
  1. $u(x) = y(-x)$
  2. $u(x) = -y(-x)$
  3. None of the above
- Note that symmetries tell you about properties of the family of solutions, not of each particular solution.

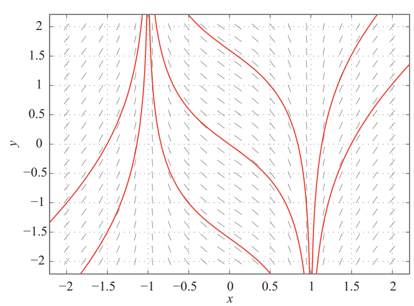
Example of application

Consider the differential equation $y' = \dfrac{x^2+1}{x^2-1}$.

- True/false: Solution curves increase for x in $(-1, 1)$.
  1. True
  2. False
- True/false: Solution curves are concave up for $x \geq 0$.
  1. True
  2. False
- True/false: Isoclines are all parallel to the y-axis.
  1. True
  2. False
- True/false: The family of solution curves is symmetric with respect to the y-axis.
  1. True
  2. False

Slope fields

A slope field for the differential equation $y' = g(x)$ is a collection of line segments in the (x, y) plane such that the slope of the segment centered at point (x, y) is equal to $g(x)$.



Slope field for the differential equation $y' = \dfrac{x^2+1}{x^2-1}$, plotted with the program PPLANE.

- Given an initial condition $y(x_0) = y_0$, one can sketch the associated solution curve (assuming it exists and is unique) by following the slope field, starting from (x_0, y_0).
- You should enter the *slope field* program into your calculator. In class, we will use PPLANE to plot slope fields.

Some of your questions

Uniqueness

- Why is it important?
- Can we see an example of a differential equation for which solutions are not unique?
- Are singularities in $g(x)$ related to uniqueness issues?
- How is uniqueness related to non-intersecting solution curves?

Some of your questions (continued)

Solutions

- Do differential equations always have an infinite number of solutions?
- If so, do they always differ by a constant?
- How do we know when a differential equation cannot be solved?
- What happens to solutions of $y' = g(x)$ as x goes to infinity?
- Can an explicit solution have a vertical tangent?
- If $g(x)$ has a vertical tangent at x_0, does it mean that if one solves $dx/dy = 1/g(x)$, one would have an horizontal tangent at that point?
- If we solve $dy/dx = g(x)$ and $dx/dy = 1/g(x)$, do we get the same solution curves?

Some of your questions (continued)

Slope fields

- How far in x and y should we go?
- How far apart should the points in the x-y plane be?
- What do isoclines mean?
- Why are slope fields useful / necessary?
- How do we identify a singularity in a slope field?
- If there exists isoclines of infinite slope, can one construct a solution that spans both sides of the isocline?
- Why can't we just use slope fields to solve ode's?