Calculus and Differential Equations I

MATH 250 A

Differential equations of the form $y' = g(y)$

Existence

$$\frac{dy}{dt} = g(y)$$

- Existence: If g is continuous on the rectangle

  $$\mathcal{R} = \{(t, y),\ |t - t_0| \leq a,\ |y - y_0| \leq b\}$$

  where $a > 0$ and $b > 0$, then there exists a continuously differentiable solution y of the above differential equation on $|t - t_0| \leq \alpha$ for which $y(t_0) = y_0$, where

  $$\alpha = \min\left(a, \frac{b}{M}\right), \qquad M = \max_{y \in [y_0 - b,\ y_0 + b]} |g(y)|.$$

  This is a simplified version of the Cauchy-Peano theorem.

Uniqueness

$$\frac{dy}{dt} = g(y)$$

- Uniqueness: If g is Lipschitz on $\mathcal{R}$, i.e. if there exists a constant $k > 0$ such that for all y_1 and y_2 in the interval $[y_0 - b,\ y_0 + b]$, we have

  $$|g(y_1) - g(y_2)| \leq k\,|y_1 - y_2|,$$

  then there exists a unique solution y to the above differential equation on $|t - t_0| \leq \alpha$, such that $y(t_0) = y_0$. The rectangle $\mathcal{R}$ and the number α are defined as in the previous theorem.

  This is a simplified version of the Picard-Lindelöf Theorem.

Examples

- Consider the differential equation $\frac{dy}{dt} = y^2$. For a given initial condition $y(t_0) = y_0$, is there always a solution that satisfies this initial condition?
  1. Yes
  2. No
- For the above differential equation, is there a unique solution going through the point $(t_0,\ y_0)$?
  1. Yes
  2. No
- For the differential equation $\frac{dy}{dt} = \sqrt{|y|}$, is there a unique solution satisfying the condition $y(3) = 0$?
  1. Yes
  2. No

Equilibrium solutions

$$\frac{dy}{dt} = g(y)$$

- A solution $y(x) = y_0$, where y_0 is such that $g(y_0) = 0$ is an equilibrium solution of the above differential equation.
- To find all equilibrium solutions, solve $g(y) = 0$ for y.
- An equilibrium solution $y = y_0$ is stable if there exists an interval $\mathcal{I} = [y_0 - \epsilon,\ y_0 + \epsilon]$, $\epsilon > 0$, such that all solutions that start in $\mathcal{I}$ converge towards y_0 as $t \to +\infty$.
- An equilibrium solution is unstable if for every $\epsilon > 0$, there always exists at least one solution starting in $\mathcal{I} = [y_0 - \epsilon,\ y_0 + \epsilon]$ that moves away from y_0 as $t \to +\infty$.

Method of solution

$$\frac{dy}{dt} = g(y)$$

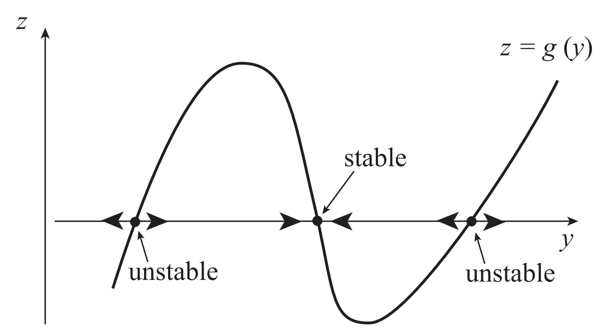
1. Re-write the equation as $\frac{1}{g(y)}\frac{dy}{dt} = 1$, and integrate both sides with respect to t.
2. This gives an equation of the form $F(y) = t + C$, where C is an arbitrary constant and F is an antiderivative of $1/g$.
3. Solution curves may therefore be obtained from one-another by translation along the t-axis.
4. As before, symmetries of g lead to symmetries of the family of solution curves.

Method of solution (continued)

$$\frac{dy}{dt} = g(y)$$

5. If possible, try to solve for y in order to obtain a family of explicit solutions.
6. Be very careful when solving for y, since it is very easy to introduce functions that are not solutions.
7. Look for singular solutions, which are solutions that are not part of the general solution given by $F(y) = t + C$. In particular, make sure you have all equilibrium solutions.
8. If given an initial or boundary condition, use it to find a particular solution. Keep in mind existence and uniqueness theorems.

Phase lines



- Equilibrium solutions of $\frac{dy}{dt} = g(y)$ are the values of y for which the graph of $g(y)$ intersects the y-axis.
- The stability of an equilibrium solution y_e may be inferred from the sign of $g(y)$ on each side of $y = y_e$.

Bifurcation diagrams

- When a parameter of the system is varied, one says that a bifurcation occurs if new solutions appear or existing solutions change stability.
- A bifurcation diagram is a plot of all equilibrium solutions as functions of a parameter of the system. Stable solutions are typically plotted with a solid stroke and unstable ones with a dashed stroke.
- The theory of dynamical systems provides a classification of bifurcations. Classical examples are
  - The pitchfork bifurcation, described by the differential equation $\frac{dy}{dt} = \mu y - y^3$.
  - The transcritical bifurcation, described by $\frac{dy}{dt} = \mu y - y^2$.
  - The saddle node bifurcation, described by $\frac{dy}{dt} = \mu - y^2$.