Calculus and Differential Equations I

MATH 250 A

Differential equations of the form $y' = g(x, y)$

Separable equations

- A first order differential equation is separable if it is of the form
$$\frac{dy}{dx} = f(y)\, g(x).$$
- To solve such an equation, we write
$$\frac{1}{f(y)}\; dy = g(x)\; dx,$$
and integrate both sides.
- Of course, when we divide by $f(y)$ we may lose equilibrium solutions, which are then singular solutions.
- Example: Solve $x^2 y' = y(y-1)$. Then, check your answer.

Equations with homogeneous coefficients

- A differential equation of the form
$$y' = g\left(\frac{y}{x}\right)$$
is said to have homogeneous coefficients.
- To solve such an equation, set $z = \frac{y}{x}$, and find an equation for z:
$$g(z) = \frac{dy}{dx} = x\frac{dz}{dx} + z.$$
- The above equation is separable and can then be solved by separation of variables. As usual, one has to be careful to keep track of singular solutions.
- Example: Solve $y' = \dfrac{x^2 + xy + y^2}{xy}$. Then, check your answer.

Linear equations

- We now consider Linear first order differential equations equations of the form
$$y' + p(x)y = q(x).$$
- To solve this equation, realize that the left-hand side can be written as
$$y' + p(x)y = \exp\left(-\int p(x)\; dx\right) \frac{d}{dx}\left(y \exp\left(\int p(x)\; dx\right)\right).$$
- As a consequence, we have
$$y(x) = \frac{1}{K(x)} \int_a^x K(t)q(t)\; dt + \frac{C}{K(x)}, \qquad K(x) = \exp\left(\int_a^x p(t)\; dt\right).$$

Linear equations (continued)

- The function $K(x)$ is called an integrating factor.
- Existence and uniqueness theorem for first-order linear equations:

  If p and q are continuous functions of x on the interval (a, b), and if $x_0 \in (a, b)$, then there is a unique solution of

  $$y' + p(x)\, y = q(x)$$

  for $x \in (a, b)$.
- Example: Solve the equation for an RL electrical circuit

  $$L\frac{dI}{dt} + R\, I = E(t),$$

  where I is the current intensity, $E(t)$ is the voltage, R is a resistance and L an inductance.

Bernoulli equations

- A first-order differential equation of the form

  $$y' + p(x)y = q(x)y^n,$$

  where n is a (positive or negative) integer with $n \neq 1$, is called a Bernoulli equation.
- To solve this equation, make the change of variable

  $$u = y^{1-n},$$

  to obtain a linear equation for u. Then, solve the equation for u and express the answer in terms of y.
- As usual when manipulating differential equations, one should be careful not to lose or introduce solutions.
- Example: Solve $y' = ay - y^3$, which is the equation describing a Pitchfork bifurcation as a goes through 0.

Existence

$$\frac{dy}{dx} = g(x, y)$$

- Existence: If g is a continuous function of x and y on the rectangle

  $$\mathcal{R} = \{(x, y),\ |x - x_0| \leq a,\ |y - y_0| \leq b\}$$

  where $a > 0$ and $b > 0$, then there exists a continuously differentiable solution y of the above differential equation on $|x - x_0| \leq \alpha$ for which $y(x_0) = y_0$, where

  $$\alpha = \min\left(a, \frac{b}{M}\right), \qquad M = \max_{(x,y)\in\mathcal{R}} |g(x, y)|.$$

  This is the Cauchy-Peano theorem.

Uniqueness

$$\frac{dy}{dx} = g(x, y)$$

- Uniqueness: If g is a continuous function of x and if there exists a constant $k > 0$ such that for all x, y_1 and y_2 in $\mathcal{R}$, we have

  $$|g(x, y_1) - g(x, y_2)| \leq k\, |y_1 - y_2|,$$

  then there exists a unique solution y to the above differential equation on $|x - x_0| \leq \alpha$, such that $y(x_0) = y_0$. The rectangle $\mathcal{R}$ and the number α are defined as in the previous theorem.

  This is the Picard-Lindelöf Theorem.