Slope fields

Consider the differential equation

$$\frac{dx}{dt} = a\,t - b\,t^2 \qquad (1)$$

where a and b are parameters.

1. For which values of t are solution curves increasing? For which values of t are they decreasing? Explain your reasoning.

2. For which values of t are solution curves concave up? For which values of t are they concave down? Explain.

3. When is the right-hand-side even? When is it odd? What symmetries do you expect for the family of solution curves in each case? Explain your reasoning.

4. Plot the slope field for $a = 1$ and $b = 1$. Use your answers to questions 1–2 above to find appropriate ranges for x and t. In particular, your plot should go far enough in the positive and negative x and t directions to include all of the salient features of the system. Paste the plot below.

5. Solve equation (1) analytically. Your answer should depend on a and b.

6. Set $a = 1$ and $b = 1$. Find the solution that goes through the point $t = 0$, $x = 1$.

7. Plot the solution you found above on the slope field (use a color other than blue) and paste the result below. Then, use DFIELD to plot the solution by clicking on the initial condition. Make sure that the numerical solution matches the solution you found.

8. Choose values of a and b for which the right-hand-side of the differential equation is odd and plot the corresponding slope field. Also use DFIELD to plot a few solution curves. Paste the output below. Does the family of solution curves have the expected symmetry? Why or why not?

9. Repeat the above question when the right-hand-side of the differential equation is even. Show all your work.

The Gompertz equation

The evolution of the number of cells N in a growing tumor is often described by the Gompertz equation

$$\frac{dN}{dt} = -a\, N \ln(b\, N), \tag{2}$$

where the parameters a and b are both positive.

1. What is the sign of N? Why?

2. For which values of N are solution curves increasing? For which values of N are they decreasing? Explain your reasoning.

3. For which values of N are solution curves concave up? For which values of N are they concave down? Explain.

4. Plot the slope field for $a = 1$ and $b = 0.1$. Use your answers to questions 1–3 above to find appropriate ranges for N and t. In particular, your plot should go far enough in the relevant N and t directions to include all of the salient features of the system. Paste the plot below.

5. Solve equation (2) analytically. Your answer (which involves two exponentials) should depend on a and b. Use your answer to find the limit of N(t) as t goes to $\pm\infty$. Explain what this means in terms of the model.

6. Set $a = 1$ and $b = 0.1$. Find the solution that goes through the point $t = 8$, $N = 6$.

7. Plot the solution you found above on the slope field (use a color other than blue) and paste the result below. Then, use DFIELD to plot the solution. Make sure that the numerical solution matches the solution you found. Note that the natural logarithm in MATLAB is denoted by `log` (and the logarithm base 10 by `log10`).

8. Use DFIELD to explore how changing a affects the slope field and the solutions. What happens if you make the substitution $x = a\, t$ in the above differential equation? Can you use this information to explain how the parameter a affects the slope field?
Hint: think in terms of comparing the graph of a function $f(t)$ with the graph of the function $f(2\, t)$.

9. Set $u = b\, N$ and find a differential equation for u. How can you use this information to explain how the parameter b affects the slope field?