Area, volume, arc length, density, & center of mass (continued)

Example 2 (continued):

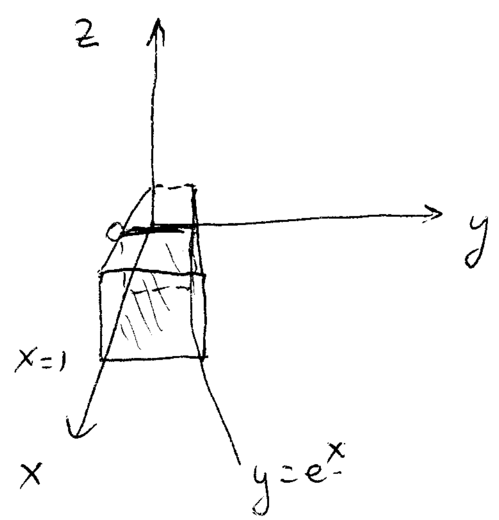
$$V = \int_0^1 \pi \left(7^2 - (7-e^x)^2\right) dx$$

$$= \pi \int_0^1 \left[49 - (49 - 14e^x + e^{2x})\right] dx$$

$$= \pi \int_0^1 (14e^x - e^{2x})\, dx = \pi \left[14e^x - \frac{1}{2}e^{2x}\right]_0^1$$

$$= \pi \left[14e - \frac{1}{2}e^2 - \left(14 - \frac{1}{2}\right)\right] \simeq 65.54$$

Example 3:

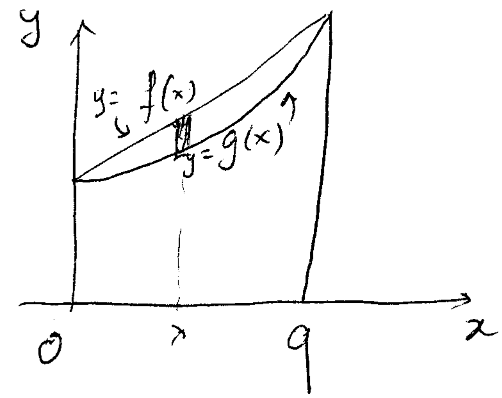


- Take slices perpendicular to the x-axis
- Area of each slice is
  $A = (e^x)^2 = e^{2x}$
- Integrate along x:

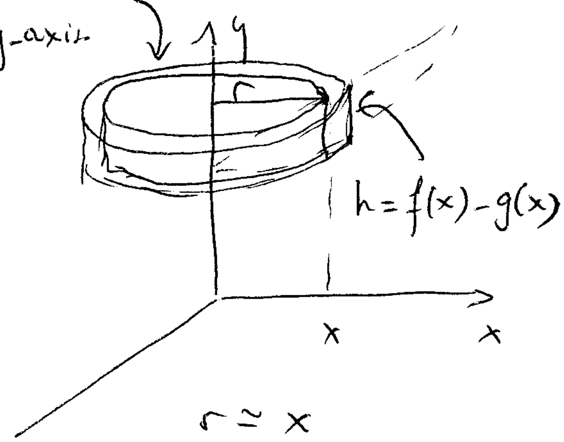
$$V = \int_0^1 e^{2x}\, dx = \left[\frac{1}{2}e^{2x}\right]_0^1$$

$$= \frac{1}{2}(e^2 - 1) \simeq 3.19$$

Example 4 :

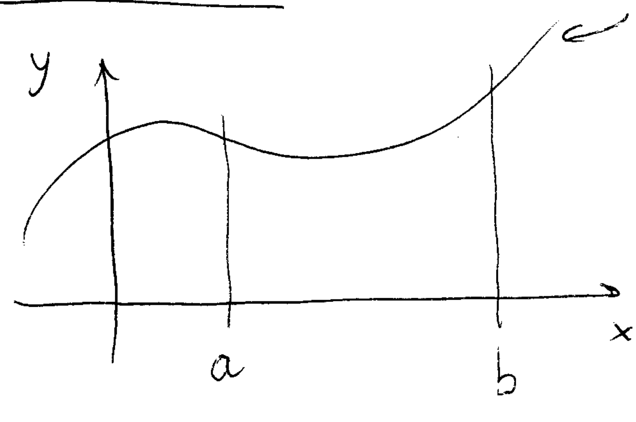


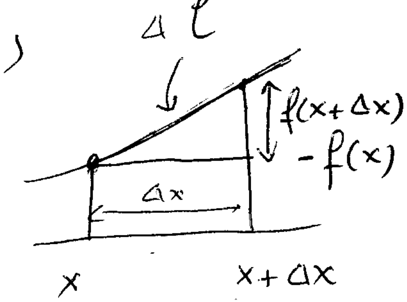
cylinder obtained by rotating slice of thickness Δx about y-axis



$$V = \int_0^9 2\pi x \left(f(x) - g(x) \right) dx$$

3. Arc length





Length $\Delta l \simeq \sqrt{(\Delta x)^2 + [f(x+\Delta x) - f(x)]^2}$

As $\Delta x \to 0$, $f(x+\Delta x) - f(x) = f'(x)\,\Delta x$

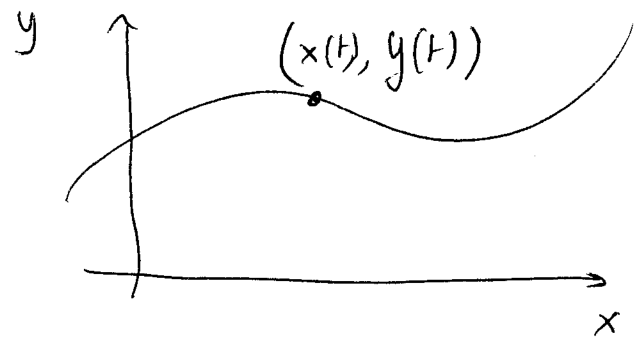
Recall $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

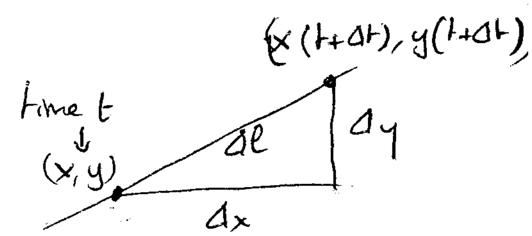
So $\Delta l \simeq \sqrt{(\Delta x)^2 + (f'(x)\,\Delta x)^2} = \Delta x \sqrt{1 + [f'(x)]^2}$

Length of curve between a & b:

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

<u>For a parametric curve</u>





$$\Delta l^2 = \Delta x^2 + \Delta y^2$$

$$\Delta x = x(t+\Delta t) - x(t) \simeq x'(t)\,\Delta t$$

<u>Recall</u> $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots$

set $x = t + \Delta t$, $a = t$: $f(t+\Delta t) = f(t) + f'(t)\,\Delta t + \frac{1}{2} f''(t)\,\Delta t^2 + \ldots$

$$\Delta y = y(t+\Delta t) - y(t) \simeq y'(t)\,\Delta t$$

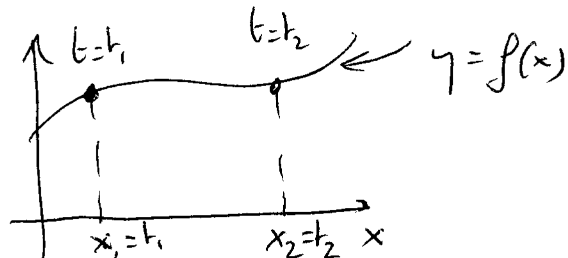
$$\Delta l^2 = (x'(t)\,\Delta t)^2 + (y'(t)\,\Delta t)^2 = [x'(t)^2 + y'(t)^2]\,\Delta t^2$$

So $\Delta l = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,\Delta t$

Length of curve (assuming velocity does not change sign (i.e. does not vanish))

$$\int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

<u>Example 1</u> : $x = t \quad y = f(t) \quad$ i.e. $\; y = f(x)$



$x'(t) = 1$

$y'(t) = f'(t)$

$$l = \int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$= \int_{t_1}^{t_2} \sqrt{1 + [f'(t)]^2}\, dt = \int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2}\, dx$$