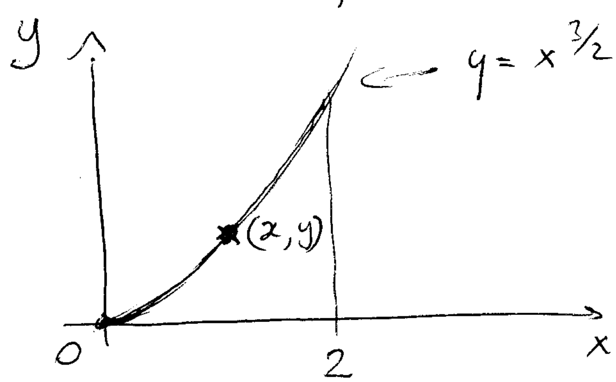
1/21/2009

Area, volume, arc length, density & center of mass (continued)

3. Arc length (continued)

Example 2: $y = x^{3/2}$ Find length of curve for $x \in [0, 2]$.

Recall: $\ell = \int_{t_1}^{t_2} \sqrt{x'(t)^2 + y'(t)^2}\, dt$



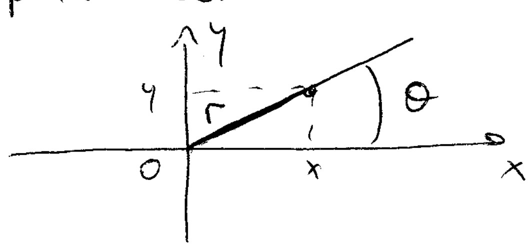
$x = t \qquad y = x^{3/2} = t^{3/2}$

Then, $\ell = \int_0^2 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\, dx$

$$\ell = \int_0^2 \sqrt{1 + \frac{9}{4}x}\, dx = \left[\frac{2}{3}\left(1 + \frac{9}{4}x\right)^{3/2} \frac{4}{9}\right]_0^2 = \frac{8}{27}\left[\left(\frac{11}{2}\right)^{3/2} - 1\right]$$

Example 3: polar coordinates

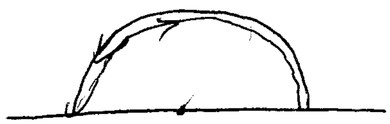
$x = r\cos(\theta)$
$y = r\sin(\theta)$



$r = \sqrt{x^2 + y^2}$

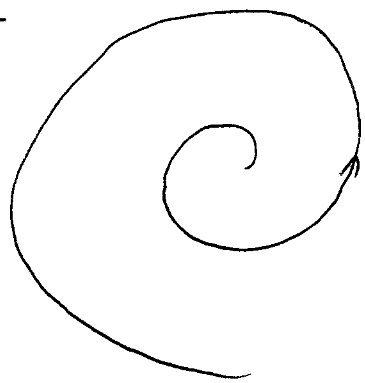
$\tan\theta = \frac{y}{x}$

Examples:

- 

$$r = 1$$
$$\theta(t) = \begin{cases} t & \text{for } 0 \leq t \leq \pi \\ 2\pi - t & \text{for } \pi \leq t \leq 2\pi \end{cases}$$

- $r = \theta$



length of spiral curve for θ between 0 & 4π

say $\quad x = r\cos(\theta) = \theta\cos(\theta)$
$\quad\quad\; y = r\sin(\theta) = \theta\sin(\theta)$

Use $\quad l = \displaystyle\int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

with $\theta = t$.

- Example 5: $\quad l(x) = \displaystyle\int_0^x \sqrt{1 + [f(s)]^2}\, ds$

$$l'(x) = \sqrt{1 + [f'(x)]^2} \quad \text{(Fundamental theorem of Calculus)}$$

$$\ell''(x) = \frac{2\,f'(x)\,f''(x)}{2\sqrt{1+[f'(x)]^2}} = \frac{f'(x)\,f''(x)}{\sqrt{1+[f'(x)]^2}}$$

So if f is increasing (i.e. $f'(x) \geq 0$), $\ell(x)$ and $f(x)$ have the same concavity.