Sequences and Series (continued)

3 Convergence of series

Convergence of geometric series

$$a + ax + ax^2 + \dots + ax^n = a\,\frac{1-x^{n+1}}{1-x}$$

If $|x|<1$ $x^{n+1} \to 0$ as $n \to \infty$

Then $$\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x}$$

If $|x|>1$, the series diverges

If $x=1$ $a + ax + ax^2 + \dots + ax^n$

$$= \underbrace{a + a + a + \dots + a}_{(n+1)\text{ terms}} = (n+1)\,a \to \infty \text{ as } n \to \infty$$

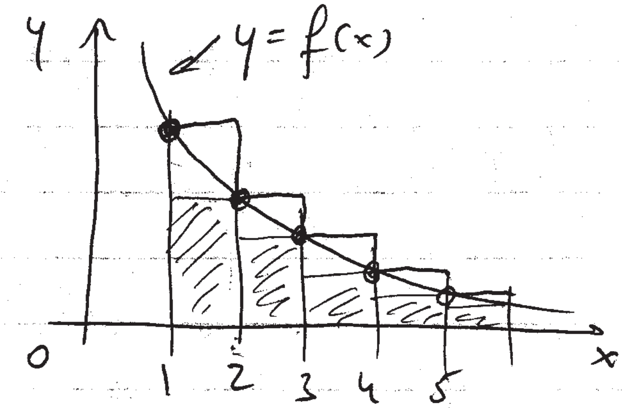
If $x=-1$ $a - a + a - a + \dots$ diverges

Properties (see slides)

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^n - 1\right] + [1] \quad \text{converges}$$

However $\sum_{n=1}^{\infty}\left[\left(\frac{1}{2}\right)^n - 1\right]$ and $\sum_{n=1}^{\infty} 1$ do not converge

Comparison with integrals



f is continuous
decreasing
positive

$u_n = f(n)$

$L = u_1 + u_2 + u_3 + \dots +$ = left hand sum approximation of $\int_1^\infty f(x)\,dx$ with $\Delta x = 1$

$R = u_2 + u_3 + u_4 + \dots + \dots =$ right hand sum approximat of $\int_1^\infty f(x)\,dx$ with $\Delta x = 1$

If $\int_1^\infty f(x)\,dx$ converges, then since f is decreasing $\int_1^\infty f(x)\,dx > R$ and the series converges.

If $\int_1^\infty f(x)\,dx$ diverges, since $0 < \int_1^\infty f(x)\,dx < L$ then the series diverges.

Example 1 $\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges because $\int_1^{\infty} e^{-x}\,dx$ converges.

Alternatively, this is a geometric series with $a = \frac{1}{e} = x$. Since $\frac{1}{e} < 1$ the series converges

So $$\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e} \frac{1}{1-\frac{1}{e}} = \frac{1}{e-1}$$

Example 2: $\sum_{n=1}^{\infty} \frac{4}{(2n+3)^3}$ converges because $\int_1^{\infty} \frac{4\,dx}{(2x+3)^3}$ converges.

Example 3: $\sum_{n=1}^{\infty} \frac{1}{1+n}$ diverges because

$$\int_1^{\infty} \frac{dx}{1+x} = \lim_{b\to\infty} \int_1^b \frac{dx}{1+x} = \lim_{b\to\infty} \left[\ln(1+x)\right]_1^b$$
$$= \infty$$

4. Tests for convergence

1. $0 \leq a_n \leq b_n$ $\qquad \sum_{n=1}^{\infty} a_n \qquad \sum_{n=1}^{\infty} b_n$

If $\sum_{n=1}^{\infty} a_n$ diverges so does $\sum_{n=1}^{\infty} b_n$

If $\sum_{n=1}^{\infty} b_n$ converges so does $\sum_{n=1}^{\infty} a_n$

Alternating series:

Assume $\lim_{n\to\infty} a_n = 0$ and $0 < a_{n+1} < a_n$

We know, that $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

$$S_n = \sum_{j=1}^{n} (-1)^{j-1} a_j .$$

$$S - S_n = \sum_{j=n+1}^{\infty} (-1)^{j-1} a_j .$$

$$= (-1)^n \left[a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots \right]$$

$$|S - S_n| = a_{n+1} - \underbrace{(a_{n+2} - a_{n+3})}_{>0} - \underbrace{(a_{n+4} - a_{n+5})}_{>0} - \cdots$$

$$< a_{n+1}$$