Sequences & series (continued)

5. Power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Assume that the above series converges when $|x-a| = R_0$, i.e. assume that

$$\sum_{n=0}^{\infty} c_n R_0^n \text{ converges}$$

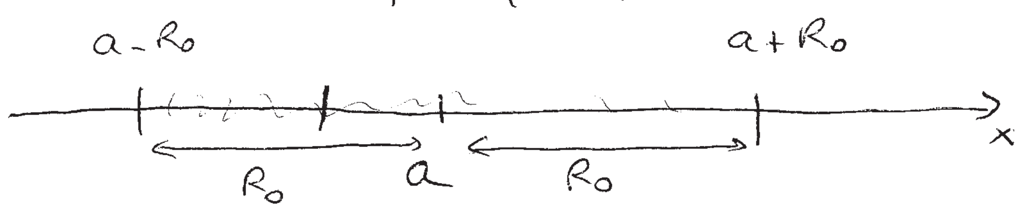
If $|x-a| < R_0$, then $|x-a|^n < R_0^n$

So $|c_n|\,|x-a|^n < R_0^n\,|c_n|$

So $0 < \sum_{n=0}^{\infty} |c_n|\,|x-a|^n < \sum_{n=0}^{\infty} |c_n|\,R_0^n$

If $\sum_{n=0}^{\infty} |c_n|\,R_0^n$ converges, then $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges absolutely, and therefore converges.

<u>Abel's lemma</u>: if the series converges for $|x-a| = R_0$ then it converge for all x's such that $|x-a| < R_0$.



<u>Example 1:</u> $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$

Think of x as given. Let $a_n = \dfrac{(x-2)^n}{n}$.

Ask when does the series converge?

Use the ratio test:

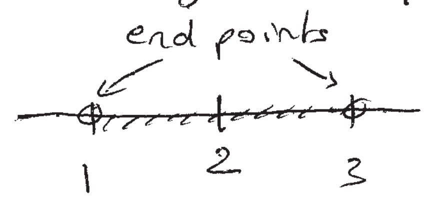
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{n+1} \cdot \frac{n}{(x-2)^n}\right| = |x-2|\left|\frac{n}{n+1}\right|$$

As $n \to \infty$ $\quad \left|\dfrac{a_{n+1}}{a_n}\right| \to |x-2|$

The series converges for $|x-2| < 1$
" diverges for $|x-2| > 1$

So the radius of convergence of the series is 1.

So the interval of convergence is given by
$|x-2| < 1 \iff 1 < x < 3$
<u>plus</u> any end point.



If $[x-2] = 1$ then

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$ diverges because of integral test.

If $x-2 = -1$ then $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

This is the harmonic series, which converges because of the alternating series test.

Therefore, the interval of convergence of $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ is $[1, 3)$.

<u>General case</u>: $a_n = c_n (x-a)^n$

Apply the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n}\right|$

i.e. $\left|\frac{a_{n+1}}{a_n}\right| = |x-a| \left|\frac{c_{n+1}}{c_n}\right|$

Assume $\left|\frac{c_{n+1}}{c_n}\right| \to L$ as $n \to \infty$ (L could be 0, or "infinity").

Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-a| L$

and $R = \frac{1}{L}$ since $L|x-a| < 1 \iff |x-a| < \frac{1}{L}$ (here L is finite).

If $L = 0$, then $\left|\frac{a_{n+1}}{a_n}\right| \to 0$ as $n \to \infty$ no matter what x is, and $R = \infty$

If $L = \infty$, then $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$ as $n \to \infty$, regardless of the value of x. So $R = 0$

Example 2: $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

This is the Taylor expansion of $\cos(x)$.

Apply the ratio test: $a_n = (-1)^n \frac{x^{2n}}{(2n)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{(-1)^n x^{2n}}\right|$$

$$= x^2 \left|\frac{(2n)!}{(2n+2)!}\right| = \frac{x^2}{(2n+2)(2n+1)}$$

$$= x^2 \left|\frac{(2n)(2n-1)(2n-2)\cdots 2\cdot 1}{(2n+2)(2n+1)(2n)(2n-1)(2n-2)\cdots 2\cdot 1}\right|$$

As $n \to \infty$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^2}{(2n+2)(2n+1)} \to 0$

So $R = \infty$, and the interval of convergence of the series is $\mathbb{R} = (-\infty, +\infty)$.

Example 3: $\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{2^n n^2}$

$$a_n = (-1)^n \frac{(x-3)^n}{2^n n^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^{n+1} \frac{(x-3)^{n+1}}{2^{n+1}(n+1)^2} \cdot \frac{2^n n^2}{(-1)^n (x-3)^n}\right|$$

$$\left|\frac{a_{n+1}}{a_n}\right| = |x-3| \ \frac{1}{2} \left(\frac{n}{n+1}\right)^2 \xrightarrow[n\to\infty]{} \frac{|x-3|}{2}$$

$$\frac{|x-3|}{2} < 1 \Leftrightarrow |x-3| < 2 \quad \text{i.e.} \quad R = 2.$$

Look at the <u>end points</u>:

If $x-3 = 2$ then we have $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{2^n n^2}$

i.e. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges because of the alternating series test.

(We also know this series is absolutely convergen because of the integral test).

If $x-3 = -2$ then we have $\sum_{n=1}^{\infty} (-1)^n \frac{(-2)^n}{2^n n^2}$

i.e. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because of the integral test.

So the interval of convergence of $\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{2^n n^2}$

is $[1, 5]$.